NATURAL SCIENCES TRIPOS Part III

Past questions

EXPERIMENTAL AND THEORETICAL PHYSICS
Minor Topics

(27 February 2010)

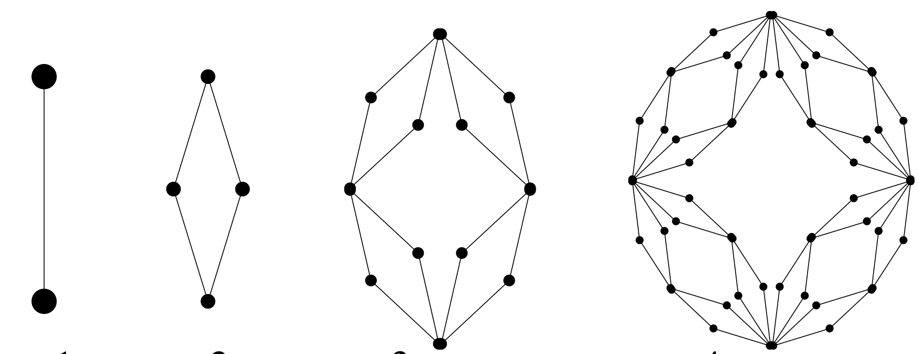
1 In one-dimension, the q-state Potts model is defined by the lattice Hamiltonian

$$\beta H = -K \sum_i \delta_{s_i, s_{i+1}} + g,$$

where the variables, s_i, can take only integer values from 1 to q, and g and $K > 0$ are constants.

(a) By integrating over the variables s_i at every other site, implement a real space renormalisation group procedure to obtain the exact recursion relations for K and g. [7]

(b) Show that $K^* = 0$ and $K^* = \infty$ are both fixed points of the Hamiltonian, and explore their stability. [6]



(c) The figure below shows the first four generations of a heirarchical lattice. Each generation, n, is obtained from generation n by replacing each bond by a 'diamond' of new bonds. Generalizing the real space renormalization group analysis above, show that, in the thermodynamic limit, $n \to \infty$, the Hamiltonian has a non-trivial fixed point defined by the condition, [5]

$$K^* = 2 \ln \left[\frac{q - 1 + e^{2K^*}}{q - 2 + 2e^{K^*}} \right].$$

Without explicit computation, sketch the resulting renormalisation group flows for the three fixed points. [2]

2 Outline concisely the conceptual basis of the *Renormalisation Group* (RG) method. [5]

In the Gaussian approximation, the Ginzburg-Landau Hamiltonian for the disordered phase of a 'smectic liquid crystal' takes the form

$$\beta H[m(\boldsymbol{x})] = \int \mathrm{d}x_\parallel \int \mathrm{d}^{d-1}\boldsymbol{x}_\perp \left[\frac{t}{2} m^2 + \frac{K}{2} (\nabla_\parallel m)^2 + \frac{L}{2} (\nabla_\perp^2 m)^2 - hm \right]$$

where $m(\boldsymbol{x})$ represents a one-component field depending on a d-dimensional set of coordinates $\boldsymbol{x} = (x_\parallel, \boldsymbol{x}_\perp)$, and the coefficients K, L, and t assume positive values.

(a) Transforming to the Fourier basis, reexpress the Hamiltonian $\beta H[m]$ in terms of the fields $m(q_{\|}, \boldsymbol{q}_{\perp})$. [3]

(b) Construct a Renormalisation Group transformation for the Hamiltonian $\beta H[m]$ by (i) applying an anisotropic rescaling of the coordinates such that $q'_{\|} = b\, q_{\|}$ and $\boldsymbol{q}'_{\perp} = c\, \boldsymbol{q}_{\perp}$, and (ii) applying the field renormalisation $m' = m/z$. How do the parameters t, K, L, and h scale under the RG transformation? [10]

(c) For what values of c and z (as a function of b) do the parameters K and L remain fixed? For the remaining coefficients t and h, show that the corresponding Gaussian fixed point is associated with the exponents $y_t = 2$ and $y_h = (d+5)/4$ respectively. [3]

(d) By establishing the relationship between the free energies $f(t,h)$ and $f(t',h')$ of the original and rescaled Hamiltonians, show that the free energy assumes the homogeneous form

$$f(t,h) = t^{2-\alpha} g_f(h/t^{\Delta}).$$

Identify the exponents α and Δ. [4]

3 In the leading approximation, the influence of lattice compressibility on the ferromagnetic transition can be explored within the framework of the Ginzburg-Landau Hamiltonian

$$\beta H[m,\phi] = \int d^3\boldsymbol{x} \left[\frac{t}{2} m^2 + u m^4 + v m^6 + \frac{K}{2} (\nabla m)^2 - hm + g\phi\, m^2 + \frac{c}{2}\phi^2 \right],$$

where $\phi(\boldsymbol{x})$ denotes the (scalar) strain field, and the parameters u and v are both assumed positive.

(a) Integrating out strain field fluctuations $\phi(\boldsymbol{x})$, show that the partition function for the magnetisation field is controlled by the effective Hamiltonian [4]

$$\beta H_{\text{eff}}[m] = \int d^3\boldsymbol{x} \left[\frac{t}{2} m^2 + \left(u - \frac{g^2}{2c} \right) m^4 + v m^6 + \frac{K}{2} (\nabla m)^2 - hm \right].$$

(b) Working in the Landau theory approximation, by sketching the m dependence of the Landau Hamiltonian for different values of the parameters, describe *qualitatively* the magnetic phase diagram for $h = 0$. In particular, discuss what happens when $\frac{g^2}{2c} > u$. [4]

(c) When $\frac{g^2}{2c} = u$, obtain the average magnetisation $\bar{m}(t, h=0)$, $\bar{m}(t=0, h)$, and susceptibility $\chi(t, h=0) = \left.\frac{\partial \bar{m}}{\partial h}\right|_{h=0}$ within the Landau theory approximation. [6]

4 If we define a Hamiltonian $\beta H[\phi] = \beta H_0[\phi] + U[\phi]$ as the sum of a free theory $\beta H_0[\phi]$ and a perturbation $U[\phi]$, show that the renormalisation group (RG) transformation resulting from field integration over fast field fluctuations $\phi_>$ results in the following renormalised Hamiltonian for the slow field fluctuations $\phi_<$,

$$\beta H'[\phi_<] = -\mathcal{Z}_>^0 + \beta H_0[\phi_<] - \ln\langle e^{-U[\phi_<,\phi_>]}\rangle_>$$

where $\mathcal{Z}_>^0 = \int D\phi_> e^{-\beta H_0[\phi_>]}$ and $\langle \cdots \rangle_> = \frac{1}{\mathcal{Z}_>} \int D\phi_> \cdots e^{-\beta H_0[\phi_>]}$. [5]

The two-dimensional sine-Gordon theory describes a free scalar field $\phi(\boldsymbol{x})$ perturbed by a periodic potential,

$$\beta H[\phi] = \int d^2\boldsymbol{x} \left[\frac{K}{2}(\nabla\phi)^2 + g\cos(\lambda\phi)\right],$$

where $K > 0$.

(a) Treating the periodic potential as a perturbation of the free Gaussian theory, and applying the perturbative momentum shell RG, show that the renormalised Hamiltonian takes the form [2]

$$\beta H'[\phi_<] = -\mathcal{Z}_>^0 + \beta H_0[\phi_<] + \int d^2\boldsymbol{x}\, g\langle\cos[\lambda(\phi_<(\boldsymbol{x}) + \phi_>(\boldsymbol{x}))]\rangle_> + \mathcal{O}(g^2)$$

(b) Working to first order in g, show that, under the RG transformation, the parameters obey the scaling relations [7]

$$\begin{cases} K(b) = Kz^2b^4 \\ g(b) = gb^2 \exp\left[-\frac{\lambda^2}{4\pi K}(1 - b^{-1})\right] \\ \lambda(b) = \zeta\lambda \end{cases}$$

[*Your discussion should indicate the significance of the parameters z, b and ζ in the RG. For a free Gaussian theory, you may assume the identity $\langle e^{i\lambda\phi(\boldsymbol{x})}\rangle = e^{-\lambda^2\langle\phi^2(\boldsymbol{x})\rangle/2}$.*]

(c) Focusing on the fixed Hamiltonian $K(b) = K$ (i.e. $z = b^{-2}$), it may be confirmed that $\lambda(b) = \lambda$. In this case, setting $b = e^\ell \simeq 1 + \ell + \cdots$, show that the differential recursion relations translate to the form

$$\frac{dg}{d\ell} = g\left(2 - \frac{\lambda^2}{4\pi K}\right).$$

Identify the fixed point and sketch the renormalisation group flow. Comment briefly on the physical implications of the result. [6]

(27 February 2010)

5 In the restricted solid-on-solid model, the Hamiltonian of a rough surface is specified by

$$H = K \sum_{\langle \boldsymbol{lm} \rangle} |h_{\boldsymbol{l}} - h_{\boldsymbol{m}}|^{\infty} ,$$

where the discrete coordinates $\boldsymbol{l}$ and $\boldsymbol{m}$ each index the sites of a two-dimensional square lattice, and the height variable $h_{\boldsymbol{l}}$ can take positive and negative integer values. Here we have used the notation $\langle \boldsymbol{lm} \rangle$ to indicate that the sum involves only neighbouring sites of the lattice.

(a) Considering βH, where $\beta = \dfrac{1}{k_{\mathrm{B}}T}$ with T the temperature, show that the height difference between neighbouring sites can only assume values of ± 1 or zero. [3]

(b) As a consequence, taking the boundary conditions to be periodic, show that the $N \times N$ site Hamiltonian may be recast in terms of the $2 \times N \times N$ variables $n_{\boldsymbol{lm}} = h_{\boldsymbol{l}} - h_{\boldsymbol{m}}$ indexing the bonds between neighbouring sites. Explain why the sum of $n_{\boldsymbol{lm}}$ around each square plaquette (i.e. unit cell boundary) of the lattice is constrained to be zero, i.e. defining $\hat{\boldsymbol{e}}_x = (1, 0)$ and $\hat{\boldsymbol{e}}_y = (0, 1)$, for each lattice site $\boldsymbol{l}$, [4]

$$n_{\boldsymbol{l},\boldsymbol{l}+\hat{\mathbf{e}}_x} + n_{\boldsymbol{l}+\hat{\mathbf{e}}_x,\boldsymbol{l}+\hat{\mathbf{e}}_x+\hat{\mathbf{e}}_y} + n_{\boldsymbol{l}+\hat{\mathbf{e}}_x+\hat{\mathbf{e}}_y,\boldsymbol{l}+\hat{\mathbf{e}}_y} + n_{\boldsymbol{l}+\hat{\mathbf{e}}_y,\boldsymbol{l}} = 0 \,.$$

(c) Imposing these constraints using the identity $\displaystyle\int_0^{2\pi} \frac{\mathrm{d}\theta}{2\pi} \mathrm{e}^{\pm in\theta} = \delta_{n0}$ for integer n, show that the partition function can be written as [6]

$$\mathcal{Z} = \left(\prod_{\boldsymbol{l}} \int_0^{2\pi} \frac{\mathrm{d}\theta_{\boldsymbol{l}}}{2\pi} \right) \exp \left\{ \sum_{\langle \boldsymbol{lm} \rangle} \ln \left[1 + 2\mathrm{e}^{-\beta K} \cos(\theta_{\boldsymbol{l}} - \theta_{\boldsymbol{m}}) \right] \right\} .$$

(d) At low temperatures (i.e. $\beta K \gg 1$), show that the system becomes equivalent to that of the classical two-dimensional XY spin model. Without resorting to detailed calculation, discuss the significance of this correspondence for the phase behaviour of the restricted solid-on-solid model? [7]

END OF PAPER

1 (a) To implement the real space renormalisation, we must integrate out spins at every other site to obtain a Hamiltonian with half the number of sites and renormalised coupling constants. Using the identity

$$\sum_{s=1}^{q} e^{K(\delta_{s_1,s}+\delta_{s,s_2})+g} = e^g \left\{ \begin{array}{ll} q-1+e^{2K} & s_1 = s_2 \\ q-2+2e^K & s_1 \neq s_2 \end{array} \right. \stackrel{!}{=} e^{K'\delta_{s_1,s_2}+g'},$$

and comparing the cases $\sigma_1 = \sigma_2$ and $\sigma_1 \neq \sigma_2$, we have

$$e^{g'} = (q-2+2e^K)e^g, \qquad e^{K'} = \frac{q-1+e^{2K}}{q-2+2e^K}.$$

(b) Setting $x = e^{K^*}$, the fixe point equation is given by

$$x = \frac{q-1+x^2}{q-2+2x}.$$

Solving this equation, we find

$$2x = -(q-2) \pm \left[(q-2)^2 + 4(q-1)\right]^{1/2} = -(q-2) \pm q,$$

i.e. $x = 1$ or $x = 1-q$. The latter solution translates to an imaginary value of k is is therefore unphysical. The former translates to $K^* = 0$. Then, for $K \ll 1$,

$$K' \simeq \ln\left[\frac{q+2K+2K^2}{q+2K+K^2}\right] \simeq \frac{K^2}{q} \ll K,$$

showing that the fixed point is stable. Conversely, for $K \gg 1$, we have

$$e^{K'} \simeq \frac{1}{2}e^K, \qquad K' = K - \ln 2 < K$$

showing that it is unstable.

(c) For the heirarchical lattice, we have

$$\left(\sum_{s=1}^{q} e^{K(\delta_{\sigma_1,s}+\delta_{s,\sigma_2})+g}\right)^2 = e^{2g} \left\{ \begin{array}{ll} (q-1+e^{2K})^2 & \sigma_1 = \sigma_2 \\ (q-2+2e^K)^2 & \sigma_1 \neq \sigma_2 \end{array} \right. \stackrel{!}{=} e^{K'\delta_{\sigma_1,\sigma_2})+g'},$$

where

$$e^{g'} = (q-2+2e^K)^2 e^{2g}, \qquad e^{K'} = \left(\frac{q-1+e^{2K}}{q-2+2e^K}\right)^2.$$

For $q = 2$, this translates to the relation

$$e^{K'} = \cosh^2(K), \qquad K' = 2\ln\cosh(K).$$

From this result, we obtain a non-trivial unstable fixed point at $K^* = 1.28$ in addition to the now stable fixed point at $K^* = \infty$ and $K^* = 0$.

(27 February 2010)

2 The divergence of the correlation length at a second order phase transition suggests that, in the vicinity of the transition, the microscopic length-scales are irrelevant. The critical behaviour is dominated by fluctuations that are statistically self-similar up to the length scale ξ. Self-similarity allows the gradual elimination of the correlated degrees of freedom at length scales $|\boldsymbol{x}| \ll \xi$, until one is left with the relatively simple uncorrelated degrees of freedom at the scale of the correlation length ξ. [5]

(a) In the Fourier representation the Hamiltonian takes the diagonal form

$$\beta H = \frac{1}{2}\int \frac{d^d\boldsymbol{q}}{(2\pi)^d} G^{-1}(\boldsymbol{q})|m(\boldsymbol{q})|^2 - hm(\boldsymbol{q}=0),$$

where the anisotropic propagator is given by [3]

$$G^{-1}(\boldsymbol{q}) = t + Kq_{\parallel}^2 + L\boldsymbol{q}_{\perp}^4.$$

(b) To implement the RG procedure, the first step is to apply a course-graining by integrating over the fast field fluctuations. Setting

$$m(\boldsymbol{q}) = \begin{cases} m_<(\boldsymbol{q}) & 0 < |q_{\parallel}| < \Lambda/b \text{ and } 0 < |\boldsymbol{q}|_{\perp} < \Lambda/c, \\ m_>(\boldsymbol{q}) & \Lambda/b < |q_{\parallel}| < \Lambda \text{ or } \Lambda/c < |\boldsymbol{q}|_{\perp} < \Lambda, \end{cases}$$

the fast fluctuations separate from the slow identically for the Gaussian Hamiltonian. As such, an integration over the fast fluctuations obtains

$$\mathcal{Z} = \mathcal{Z}_> \int Dm_< \exp\left[-\frac{1}{2}\int_0^{\Lambda/b}(\mathrm{d}q_{\parallel})\int_0^{\Lambda/c}(\mathrm{d}^{d-1}\boldsymbol{q}_{\perp})G^{-1}(\boldsymbol{q})|m_<(\boldsymbol{q})|^2 + hm_<(0)\right],$$

where the constant $\mathcal{Z}_>$ is obtained from performing the functional integral over $m_>$. Applying the rescaling $q'_{\parallel} = bq_{\parallel}$ and $\boldsymbol{q}'_{\perp} = c\boldsymbol{q}_{\perp}$, the cut-off in the domain of momentum integration is restored. Finally, applying the renormalisation $m'(\boldsymbol{q}) = m_<(\boldsymbol{q})/z$ to the Fourier field amplitudes, one obtains

$$\mathcal{Z} = \mathcal{Z}_> \int Dm'(\boldsymbol{q}')e^{-(\beta H)'[m'(\boldsymbol{q}')]},$$

where the renormalised Hamiltonian takes the form

$$(\beta H)' = \frac{1}{2}\int (\mathrm{d}^d\boldsymbol{q})b^{-1}c^{-(d-1)}z^2\left(t + Kb^{-2}{q'_{\parallel}}^2 + Lc^{-4}{\boldsymbol{q}'_{\perp}}^4\right)|m'(\boldsymbol{q}')|^2 - zhm'(0).$$

From the result, we obtain the renormalisation of the coefficients [10]

$$\begin{cases} t' = tb^{-1}c^{-(d-1)}z^2, \\ K' = Kb^{-3}c^{-(d-1)}z^2, \\ L' = Lb^{-1}c^{-(d+3)}z^2, \\ h' = hz. \end{cases}$$

(c) Choosing parameters $c = b^{1/2}$ and $z = b^{(d+5)/4}$ ensures that $K' = K$ and $L' = L$ and implies the scaling exponents $y_t = 2$, $y_h = (d+5)/4$. [3]

(d) From this result we obtain the renormalisation of the free energy density

$$f(t,h) = b^{-(d+1)/2} f(b^2 t, b^{(d+5)/4} h).$$

Setting $b^2 t = 1$, we can identify the exponents $2 - \alpha = (d+1)/4$ and $\Delta = y_h/y_t = (d+5)/8$. [4]

3 The Hamiltonian given in the question represents the canonical form of the Ginzburg-Landau Hamiltonian for a second order phase transition. In the Landau theory, the functional integral for the classical partition function is approximated by its value at the Hamiltonian minimum, viz. [2]

$$\mathcal{Z} \equiv e^{-\beta F} = \int Dm(\boldsymbol{x}) e^{-\beta H[m(\boldsymbol{x})]} \simeq \exp\left[-\min_{m(\boldsymbol{x})} \beta H[m(\boldsymbol{x})]\right].$$

For $K > 0$, the minimal Hamiltonian is given by $m(\boldsymbol{x}) = \bar{m}$, constant. In this approximation, the free energy density is given by $f = \frac{\beta F}{V} = \beta H[\bar{m}]$, where $\bar{m} + 4u\bar{m}^3 - h = 0$. In particular, for $h = 0$, the magnetisation acquires a non-zero expectation value when $t < 0$ with $\bar{m} = \sqrt{-t/4u}$. Similarly, for $t = 0$, the magnetisation varies as $m = (h/3u)^{1/3}$. From this result, one can infer a phase diagram in which a line of first order transitions along $h = 0$ terminates at the critical point $t = 0$. Finally, differentiating the condition on $\bar{m}$ with respect to $\bar{m}$, one obtains the susceptibility [4]

$$\chi(t, h=0) = \left.\frac{\partial m}{\partial h}\right|_{h=0} = \begin{cases} 1/t & t > 0 \\ -1/2t & t < 0. \end{cases}$$

[*Full credit will be given even if the specific heat is not derived.*]

(a) In the presence of the strain field, the partition function is given by

$$\mathcal{Z} = \int Dm\, D\phi\, e^{-\beta H[m,\phi]}.$$

Being Gaussian in ϕ, the integral may be performed exactly and obtains

$$\int D\phi e^{-\int d^3\boldsymbol{x}\left[\frac{c}{2}\phi^2 + g\phi m^2\right]} = \int D\phi e^{-\int d^3\boldsymbol{x}\left[\frac{c}{2}(\phi - \frac{gm^2}{c})^2 - \frac{g^2}{2c}m^4\right]} = \text{const.} \times e^{\int d^3\boldsymbol{x}\left[\frac{g^2}{2c}m^4\right]},$$

leading to the suggested reduction in the quartic coefficient. [4]

(b) While the quartic coefficient $u' = u - g^2/2c$ remains positive, the Landau theory continues to predict a second order transition at $h = t = 0$. However, when the sign is reversed, the Landau Hamiltonian $(h = 0)$

$$\psi(m) = \frac{t}{2}m^2 + u'm^4 + vm^6$$

develops additional minima. By sketching $\psi(m)$ for different parameter values, one may see that, for $u' < 0$ and $t = 0$ the (degenerate) global minimum lies at some non-zero value of $\bar{m}$ while, for t large, the global minimum lies a $\bar{m} = 0$. In between, there exists a line of first order transitions which merges with the line of second order critical points at the tricritical point $t = u' = 0$. [*By careful calculation, one may show that the first order boundary follows the line* $t = u'^2/2v$.] [4]

(c) Near the tricritical point ($u' = 0$ and $h = 0$), one obtains

$$\frac{\partial \psi}{\partial m} = m(t + 6vm^4) = 0, \qquad \bar{m}(t, h = 0) = \begin{cases} 0 & t > 0, \\ (-t/6v)^{1/4} & t < 0, \end{cases}$$

implying an exponent $\beta = 1/4$. Similarly, for $t = 0$, one obtains

$$h = 6v\bar{m}^5, \qquad \bar{m}(t = 0, h) = (h/6v)^{1/5}$$

i.e. $\delta = 5$. Finally, for finite h and t, differentiating the defining equation for $\bar{m}$, one obtains the susceptibility [4]

$$\chi(t, h = 0) = \left.\frac{\partial \bar{m}}{\partial h}\right|_{h=0} = (t + 30v\bar{m}^4)^{-1},$$

implying that $\chi \sim 1/|t|$ for $t < 0$ and $t > 0$. Thus we find the exponent $\gamma = 1$. [2]

4 Separating the field fluctuations into fast and slow degrees of freedom, $\phi(\boldsymbol{x}) = \phi_<(\boldsymbol{x}) + \phi_>(\boldsymbol{x})$,

$$\begin{aligned} \mathcal{Z} &= \int D\phi_< e^{-\beta H_0[\phi_<]} D\phi_> e^{-\beta H[\phi_>] - U[\phi_<, \phi_>]} \\ &= \mathcal{Z}_>^0 \int D\phi_< e^{-\beta H_0[\phi_<]} \langle e^{-U[\phi_<, \phi_>]} \rangle_> \\ &= \mathcal{Z}_>^0 \int D\phi_< e^{-\beta H_0[\phi_<]} + \ln\langle e^{-U[\phi_<, \phi_>]} \rangle_>. \end{aligned}$$

From this result, one obtains the required renormalised Hamiltonian. [5]

(a) Applying the perturbative expansion,

$$-\ln\langle e^{-U[\phi_<, \phi_>]} \rangle_> \simeq \langle U[\phi_<, \phi_>] \rangle_> + O(U^2).$$

to the sine-Gordon theory, one obtains the required expression for the Hamiltonian. [2]

(b) Integrating over the fast field fluctuations, [2]

$$\begin{aligned}\langle g\cos[\lambda(\phi_<(\boldsymbol{x})+\phi_>(\boldsymbol{x}))]\rangle_> &= g\mathrm{Re}\left[e^{i\lambda\phi_<(\boldsymbol{x})}\langle e^{i\lambda\phi_>(\boldsymbol{x})}\rangle_>\right]\\ &= ge^{-\lambda^2\langle\phi_>^2(\boldsymbol{x})\rangle/2}\cos(\lambda\phi_<(\boldsymbol{x}))\end{aligned}$$

Then, making use of the identity, [2]

$$\langle\phi_>^2(\boldsymbol{x})\rangle = \int_> \frac{d^2\boldsymbol{q}}{(2\pi)^d}\frac{1}{Kq^2} = \frac{1}{2\pi K}(1-b^{-1})$$

and applying the rescalings, [2]

$$\boldsymbol{q}' = \boldsymbol{q}/b, \qquad \phi'(\boldsymbol{q}') = \phi_<(\boldsymbol{q})/z, \qquad \phi'(\boldsymbol{x}') = \phi_<(\boldsymbol{x})/\zeta,$$

one obtains the renormalised Hamiltonian [1]

$$\beta H'[\phi'] = \int \frac{d^2\boldsymbol{q}}{(2\pi)^2}\frac{K(b)}{2}\boldsymbol{q}'^2|\phi(\boldsymbol{q})|^2 + \int d^2\boldsymbol{x}'\, g(b)\cos\left[\lambda(b)\phi'(\boldsymbol{x}')\right]$$

where the coefficients are as stated.

(c) Using the expansion, [3]

$$\begin{aligned}g(\ell) &= g(0)e^{2\ell}\exp\left[-\frac{\lambda^2}{4\pi K}(1-e^{-\ell})\right]\\ g(0)+\ell\frac{dg}{d\ell}+\cdots &= g(0)\left[1+2\ell-\frac{\lambda^2}{4\pi K}\ell+\cdots\right]\end{aligned}$$

one recovers the required differential recursion relation. For $\lambda^2 > 8\pi K$, $g(\ell)$ dimishes under the RG and the system flows towards a free massless theory. Conversely, for $\lambda^2 < 8\pi K$, $g(\ell)$ grows under RG leading to a confined or massive theory. When $\lambda_*^2 = 8\pi K$, the Hamiltonian is fixed and the theory critical. [3]

5 (a) For $h_i = h_j$ the site energy of a link $\beta H_{ij} = 0$; for $h_i = h_j \pm 1$ $\beta H_{ij} = K$; and $\beta H_{ij} \to \infty$ otherwise. Therefore, the former $n_{ij} = h_i - h_j = 0, \pm 1$ are the only allowed field configurations. [3]

(b) Taking into account the constraint $n_{ij} = 0, \pm 1$, one may note that the sum of n_{ij} around a plaquette $\sum_{\mathrm{ij}\in\mathrm{plaquette}} n_{ij} = 0$. Such a constraint ensures that the sum of n_{ij} around any closed loop must vanish since any loop can be decomposed into a set of elementary plaquettes. [4]

(c) Then, making use of the identity given in the question to impose the constraint, the partition function may be written as [6]

$$\mathcal{Z} = \sum_{n_{ij}=0,\pm 1} e^{-K|n_{ij}|}\left(\prod_{\boldsymbol{i}}\int_0^{2\pi} d\theta_{\boldsymbol{i}} e^{i(n_{i,i+\hat{e}_x}+n_{i+\hat{e}_x,i+\hat{e}_x+\hat{e}_y}+n_{i+\hat{e}_x+\hat{e}_y,i+\hat{e}_y}+n_{i+\hat{e}_y,i})\theta_i}\right),$$

where the product runs over all lattice sites $\boldsymbol{i}$. Noting that each site $\boldsymbol{i}$ is associated with two bonds along direction $\hat{\boldsymbol{e}}_x$ and $\hat{\boldsymbol{e}}_y$, the partition function may be rearranged as

$$\begin{aligned}\mathcal{Z} &= \left(\prod_{\boldsymbol{i}}\int_0^{2\pi} d\theta_{\boldsymbol{i}}\right)\left[\sum_{n=0,\pm1} e^{-K|n|}e^{i(\theta_{\boldsymbol{i}}+\theta_{\boldsymbol{i}-\hat{\boldsymbol{e}}_y})n}\right]\left[\sum_{n=0,\pm1} e^{-K|n|}e^{i(\theta_{\boldsymbol{i}}+\theta_{\boldsymbol{i}-\hat{\boldsymbol{e}}_x})n}\right]\\ &= \left(\prod_{\boldsymbol{i}}\int_0^{2\pi} d\theta_{\boldsymbol{i}}\right)\exp\left[\ln(1+2e^{-K}\cos(\theta_{\boldsymbol{i}}+\theta_{\boldsymbol{i}-\hat{\boldsymbol{e}}_y}))+\ln(1+2e^{-K}\cos(\theta_{\boldsymbol{i}}+\theta_{\boldsymbol{i}-\hat{\boldsymbol{e}}_x}))\right].\end{aligned}$$

Finally, setting $\theta_{\boldsymbol{i}} \mapsto -\theta_{\boldsymbol{i}}$ on alternate lattice sites, one obtains

$$\mathcal{Z} = \left(\prod_{\boldsymbol{i}}\int_0^{2\pi} d\theta_{\boldsymbol{i}}\right)\exp\left[\sum_{\langle\boldsymbol{ij}\rangle}\ln(1+2e^{-K}\cos(\theta_{\boldsymbol{i}}-\theta_{\boldsymbol{j}}))\right].$$

(d) At low temperatures ($K \gg 1$), the logarithm may be expanded as

$$\mathcal{Z} = \left(\prod_{\boldsymbol{i}}\int_0^{2\pi} d\theta_{\boldsymbol{i}}\right)\exp\left[2e^{-K}\sum_{\langle\boldsymbol{ij}\rangle}\cos(\theta_{\boldsymbol{i}}-\theta_{\boldsymbol{j}})\right].$$

The latter can be identified as the partition function of a two-dimensional XY model with exchange constant $J = 2e^{-K}$. This correspondence allows us to infer that the proliferation of massless fluctuations of the fields $\theta_{\boldsymbol{i}}$ leads to a disordering of the system for any non-zero temperature, i.e. spatial correlations of the height degrees of freedom allow for divergent fluctuations. However, since the present system lies at the lower critical dimension, one can infer that the restricted solid on solid model exhibits a topological Kosterlitz-Thouless phase transition from a phase with power-law correlations of the order parameter to exponential correlations. [7]