Density Waves and Supersolidity in Rapidly Rotating Atomic Fermi Gases

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(Received 29 April 2007; published 9 November 2007)

We study theoretically the low-temperature phases of a two-component atomic Fermi gas with attractive s-wave interactions under conditions of rapid rotation. We find that, in the extreme quantum limit, when all particles occupy the lowest Landau level, the normal state is unstable to the formation of charge density wave (CDW) order. At lower rotation rates, when many Landau levels are occupied, we show that the low-temperature phases can be supersolids, involving both CDW and superconducting order.

DOI: 10.1103/PhysRevLett.99.190409 PACS numbers: 05.30.Fk, 03.75.Hh, 03.75.Ss, 67.90.+z

The experimental achievement of condensation of pairs of atoms in two-component Fermi gases with resonant s-wave interactions [1–4] has allowed studies of interacting Fermi systems in regimes not accessible in solid-state systems: notably the transition from a BCS superfluid to a Bose-Einstein condensate and regimes of large density imbalance between the two species. The ability to rotate the gases, revealing a lattice of quantized vortices [5], has provided an important diagnostic of superfluidity in these phase-coherent condensates.

A very interesting regime arises in atomic Fermi gases under conditions of rapid rotation (high vortex density). Noting the analogy between rotation and magnetic field in a superconductor, one might anticipate the BCS phase to revert to a normal state above a critical rotation frequency (analogous to H_{c2} in superconductivity), as predicted by BCS theory within a semiclassical approximation [6–8]. Yet, going beyond this approximation to include Landau level (LL) structure, one finds that the normal phase can be unstable to ordered phases involving high-field superconductivity (SC) [9], "charge" density wave (CDW), or spin-density wave (SDW) order [10].

In this Letter, we investigate the low-temperature phases of a two-component atomic Fermi gas with attractive s-wave interactions under conditions of rapid rotation. The regime of interest for atomic gases differs substantially from regimes studied in solid-state systems: the rotation does not lead to any "Zeeman" splitting which might suppress high-field SC order; the short-range interactions allow density wave order to develop (this is suppressed in solid-state systems by Coulomb interactions). We show that the low-temperature phases of an atomic Fermi gas with attractive interactions involve an interesting interplay between CDW and superconducting phases. In the extreme quantum limit, when only the lowest Landau level (LLL) is occupied, we show that the system is unstable to CDW order along the rotation axis. At lower rotation rates, we show that CDW and SC can coexist, leading to "supersolid" behavior.

We study a rapidly rotating gas of two-species fermions, of equal densities, in the uniform limit: the number of vortices is assumed large, so the rotation frequency, Ω, is close to the trap frequency, and the confinement along the rotation axis is assumed weak. Our results for the infinite system can be used to infer the properties of a finite cloud within the local density approximation. In the rotating frame, the Coriolis force mimics a magnetic field and leads to a Landau level structure with cyclotron frequency $\omega_c = 2\Omega$. The single particle states then have energies $\epsilon_\nu = (2n+1)\hbar\Omega + \frac{\hbar^2 k^2}{2m}$, where $\nu = (n, x, k)$ stands for the LL index n, the momentum in the Landau gauge x [11], and the wave vector along the rotation axis k. For a noninteracting gas with Fermi energy ϵ_F, the nth Landau level has a 1D Fermi surface with Fermi momentum $\hbar k_{Fn} = \{2m[\epsilon_F - (2n+1)\hbar\Omega]\}^{1/2}$ and kinetic energy relative to the bottom of the band $\epsilon_{Fn} = \hbar^2 k_{Fn}^2/2m$. We describe the instabilities of these Fermi surfaces arising from weak interactions. (We focus on results for attractive interactions but also report on the repulsive case.)

First, we analyze the effect of rotation on the SC phase, applying BCS theory in the presence of Landau level structure [6,7,9,13–15]. For contact interactions, the gap equation requires regularization at high energies. For solid-state systems, the Debye frequency provides a natural cutoff for phonon-mediated attractive interactions. In a cold atomic gas, a natural regularization arises from the (small) length scale of the interparticle forces. Using a two-channel model for the Feshbach interaction, this length scale enters as the size of the "closed channel" boson (see e.g. [8,16,17]) and can be taken to zero with the introduction of appropriate counterterms. Following Ref. [16], the parameters of the model are the boson energy $\epsilon_B = \hbar\Omega + \delta + C$ and the coupling $\alpha S_{\nu\nu'}$ between a closed channel boson and fermions with quantum numbers ν and ν'. Here, δ is the physical detuning of the bosons and C a counterterm which is set to cancel the boson self energy $\Sigma(\omega \to 0) = \alpha^2 \sum_{\nu\nu'} |S_{\nu\nu'}|^2/(\epsilon_\nu + \epsilon_{\nu'})$, such that the model reproduces the scattering properties at low energy and $\Omega \to 0$ [16,18]. The physical scattering parameters are related via $-\alpha^2/\delta = 4\pi\hbar^2 a_s/m \equiv g$. Treating the ensuing two-channel Hamiltonian within mean field, and assuming a wide Feshbach resonance, yields the linearized

gap equation [16]

$$\frac{1}{-a_s} = \hbar\Omega \sum_{n,n'=0}^{\infty} B_n^{n'} \int \frac{dk}{2\pi} \left[\frac{\mathrm{th}\frac{\xi_\nu}{2k_BT} + \mathrm{th}\frac{\xi_{\nu'}}{2k_BT}}{\xi_\nu + \xi_{\nu'}} - \frac{2}{\epsilon_\nu + \epsilon_{\nu'}} \right], \tag{1}$$

with $B_n^{n'} = \binom{n+n'}{n} 2^{-n-n'}$, $\xi_\nu = \epsilon_\nu - \mu$, and the magnetic length $\ell_0 \equiv (\hbar/2m\Omega)^{1/2}$. The solutions to (1) determine the critical temperature T_c for superconductivity.

Within a semiclassical approximation to (1), T_c vanishes for $\hbar\Omega \gtrsim \Delta^2/\mu$ (Δ is the zero field gap) [6–8,19]. The full gap Eq. (1) admits solutions even in this regime. Then, when T_c is small, the dominant contributions arise from integrating the "diagonal" terms ($n = n'$) [14], which diverge logarithmically at low T for occupied LLs. Provided $k_BT_c \ll [\mu - \hbar\Omega(2n_{\max} + 1)]$, the off-diagonal terms ($n \neq n'$) can be neglected, and one finds

$$T_c \sim \eta \frac{\hbar\Omega}{k_B} \exp\left\{ -\frac{2\pi}{-a_s k_{F0}} G(\eta)^{-1} \right\}, \tag{2}$$

where $\eta \equiv (\mu - \hbar\Omega)/(2\hbar\Omega)$, $n_{\max} = \lfloor \eta \rfloor$, and

$$G(\eta) \equiv \frac{1}{\eta} \sum_{n=0}^{n_{\max}} \frac{(2n)!}{(2^n n!)^2} \left(1 - \frac{n}{\eta}\right)^{-1/2}. \tag{3}$$

The critical temperature (2) is a strongly oscillating function of $\mu/\hbar\Omega$, with a peak each time a LL depopulates and $G(\eta)$ diverges. The sharp peaks predicted by (2) are rounded in a full of solution of (1) which is required for strong coupling. The evolution from weak to strong coupling is shown in Fig. 1, which we have computed by solving (1) using a numerical root-finding routine.

Consistent with previous studies of BCS theory in solid-state systems [6,7,9,13–15], we find that LL quantization leads to a stable SC state at any value of the field [9]. For

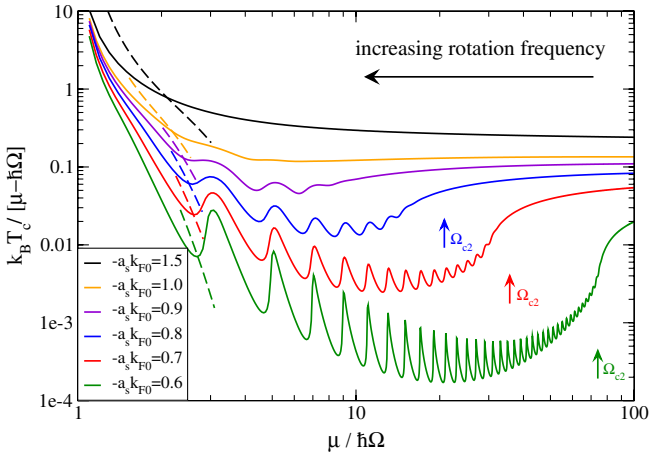

FIG. 1 (color online). The critical temperature calculated within BCS mean field, for rotation frequency Ω and chemical potential μ. Arrows indicate critical frequencies Ω_{c2} to the left of which T_c vanishes in the semiclassical approximation [7]. Dashed lines show T_c of the CDW state in the LLL as obtained in the parquet approximation.

$|a_s|k_F \lesssim 1$ the critical temperature has a minimum value $T_c^{\min}(a_s k_F)$. For temperatures $T \gtrsim T_c^{\min}(a_s k_F)$, mean-field theory results predict a series of reentrant SC to normal transitions as the rotation rate increases. Our results differ from those presented in Ref. [16]: the (reentrant) superconductivity at rapid rotation was not found in that work; the critical rotation frequency has an important temperature-dependence, especially for strong coupling.

The superconducting phase competes with other ordered phases. To determine the nature of the ground state one must work beyond mean-field theory. We analyze the competition between SC and other ordered phases within a single LL, for example when all particles occupy the LLL. Owing to the quasi-1D dispersion within this LL, any response function that connects opposite sides of the Fermi surface diverges at low temperatures as a power of

$$\xi = \frac{|g|}{(2\pi)^3 \hbar v_{Fn} \ell_0^2} \ln\left(\frac{\epsilon_{Fn}}{k_BT}\right). \tag{4}$$

Identical divergences occur in both particle-particle (p-p) and particle-hole (p-h) diagrams [20] and in diagrams of higher orders. The resulting ensemble of "parquet" diagrams, obtained by mutual insertion of p-p and p-h blocks into one another [21], is most easily analyzed in terms of a renormalization group (RG) approach [12]. This scheme has been applied to spinless electrons in a magnetic field [12]. We generalize this approach to a two-component rotating atomic Fermi gas and consider particles in the nth LL. Particles at the two Fermi points $k = \pm k_{Fn}$ are represented by separate fermionic field operators $\hat{a}^{(\dagger)}$ and $\hat{b}^{(\dagger)}$. States at the first Fermi point are expanded in terms of the LL wave functions [11]

$$\psi_{nxk}(X, Y, Z) = \mathcal{N}_n H_n(X - x) e^{ixY + (X-x)^2/2 + ikZ}, \tag{5}$$

with lengths measured in units of ℓ_0, the Hermite polynomials H_n, and normalization $\mathcal{N}_n = (L_y L_z \pi^{\frac{1}{2}} 2^n n!)^{-1/2}$. At the other Fermi point we use the transformed basis

$$\tilde{\psi}_{nyk}(X, Y, Z) = \frac{1}{N_\phi} \sum_x e^{-ixy} \psi_{nxk}(X, Y, Z). \tag{6}$$

For weak coupling, the kinetic energy can be linearized around the Fermi points, $\pm k_{Fn}$. The (logarithmically divergent) part of the contact interaction (amplitude g) describing scattering between opposite Fermi surfaces is

$$\begin{aligned} \mathcal{H}_I = \frac{g}{L_x L_y L_z} \sum_{\mu\nu\sigma\rho} \sum_{\substack{k_1,k_2,k_3 \\ x,x',y,y'}} (\delta_{\mu\rho}\delta_{\nu\sigma} - \delta_{\mu\sigma}\delta_{\nu\rho}) e^{i(xy' - x'y)} \\ \times \gamma_0^{(n)}(x - x', y - y') \\ \times \hat{a}^\dagger_{n,x,k_1,\mu} \hat{b}^\dagger_{n,y,k_2,\nu} \hat{b}_{n,y',k_3,\rho} \hat{a}_{n,x',k_1+k_2-k_3,\sigma}. \end{aligned} \tag{7}$$

The dependence on the LL index n arises only in the form of the bare interaction vertex, $\gamma_0^{(n)}(\mathbf{r}) = e^{-(1/2)r^2}[L_n(r^2/2)]^2$, where we introduce $\mathbf{r} \equiv (x, y)$ and L_n are the Laguerre polynomials. The interaction $\mathcal{H}_I$

can be viewed as two distinct vertices according to the way spin is conserved and is denoted $\gamma_{1,2}$ in the usual notation for quasi-1D systems [22]. From (7), these vertices have the initial conditions

$$\gamma_{1,2}^{(n)}(\mathbf{r})|_{\xi=0} = \mathrm{sgn}(g)\gamma_0^{(n)}(\mathbf{r}). \tag{8}$$

Renormalization of the vertices $\gamma_{1,2}$ leads to corrections that can be expressed as a power in ξ (4) [21]. The one-loop RG equations can be obtained by adapting the approach of Ref. [23] to include the LL structure. We find

$$\frac{d\gamma_1}{d\xi} = -2\gamma_1 * \gamma_1 + 2\gamma_1 * \gamma_2 - 2\gamma_1 \otimes \gamma_2 \tag{9}$$

$$\frac{d\gamma_2}{d\xi} = 2\gamma_2 * \gamma_2 - \gamma_1 \otimes \gamma_1 - \gamma_2 \otimes \gamma_2, \tag{10}$$

where the operations $*$ and $\otimes$ arise in p-h and p-p loops, respectively, and are defined by

$$\equiv \gamma_i * \gamma_j \equiv \int d^2\mathbf{r}'\gamma_i(\mathbf{r}-\mathbf{r}')\gamma_j(\mathbf{r}'). \tag{11}$$

$$\equiv \gamma_i \otimes \gamma_j \equiv \int d^2\mathbf{r}'\gamma_i(\mathbf{r}-\mathbf{r}')\gamma_j(\mathbf{r}')e^{-i\mathbf{r}\wedge\mathbf{r}'}. \tag{12}$$

The phase factor in (12) is a consequence of the LL structure.

We have solved the RG Eqs. (9) and (10) with initial conditions (8) for arbitrary Landau level index n, using a standard numerical routine with $\gamma_{1,2}(|\mathbf{r}|)$ discretized uniformly in $|\mathbf{r}|$. (The initial conditions for $\gamma_{1,2}$ are radially symmetric, and this symmetry is preserved by the RG equations.) To identify instabilities, we calculate the renormalization of the response functions [23]. The RG equations for the triangular vertices $\mathcal{T}$ in the (singlet) SC, charge and spin-density wave (SDW) channels are given in our case by

$$d_\xi \mathcal{T}_{\mathrm{SSC}} = (-\gamma_1 - \gamma_2) \otimes \mathcal{T}_{\mathrm{SSC}} \tag{13}$$

$$d_\xi \mathcal{T}_{\mathrm{CDW}} = (-2\gamma_1 + \gamma_2) * \mathcal{T}_{\mathrm{CDW}} \tag{14}$$

$$d_\xi \mathcal{T}_{\mathrm{SDW}} = \gamma_2 * \mathcal{T}_{\mathrm{SDW}}. \tag{15}$$

Initial conditions for the triangular vertices can be chosen as $\mathcal{T}_i|_{\xi=0} = \delta(\mathbf{r})$, such that all Fourier components are nonzero. We find the smallest value, ξ_c, at which a susceptibility diverges: this indicates a transition into an ordered phase at a critical temperature [see (4)]

$$T_c \sim \frac{\epsilon_{Fn}}{k_B} \exp\left(-\frac{(2\pi)^3\hbar v_{Fn}\ell_0^2}{|g|}\xi_c\right). \tag{16}$$

In contrast to the full RG equations, the simplified equations describing only p-h ladders can be solved analytically and provide a useful reference point for our numerical evaluation. The solution for the p-h ladder discussed in Ref. [12] can be generalized to arbitrary LL index n and yields a transition at a critical temperature that is independent of n and the sign of g, with $\xi_c = (2\pi)^{-1}$. [For $g < 0$ ($g > 0$) the transition is to a CDW (SDW).] For p-p ladders, the problem can be solved analytically for $n = 0$, where the SC instability occurs for attractive interactions also at $\xi_c = (2\pi)^{-1}$. By restricting the SC gap equation in the presence of a magnetic field to a single LL (see above and [14]), one can infer $\xi_c(n) = (2^n n!)^2/[2\pi(2n)!]$, showing that SC order becomes weak as $n \to \infty$. These analytic results are reproduced by our numerical approach, when restricted to include p-p or p-h diagrams only. Note that for $n = 0$ the CDW and SC instabilities have the same critical temperature. Thus, mean-field theory cannot determine which of these states will form the low-temperature phase.

Our solution of the full RG Eqs. (9) and (10) shows that, for attractive interactions, CDW order is the dominant instability for all LLs. The critical temperature (16) depends on the LL index, with exponents summarized in Table I. Thus, for the LLL $n = 0$, the competition between the identical instabilities in p-p and p-h channels [both at $\xi_c = (2\pi)^{-1}$] is decided to the advantage of CDW order. The order parameter diverges most strongly at zero in-plane momentum, so the density waves are aligned with the rotation axis. Thus, the CDW phase in the nth LL involves a modulation of the particle density along the rotation axis, with period $\lambda_n^{\mathrm{CDW}} = \pi/k_{Fn}$. Within one period of the density wave the effective 2D particle density (in that LL) is $n_{2d,n} = 1/(\pi\ell_0^2)$, such that this LL is fully occupied: its filling factor is $\nu_n \equiv n_{2d,n}2\pi\ell_0^2 = 2$. Thus the CDW phase is fully gapped. In the extreme quantum limit, when $\hbar\Omega < \epsilon_F < 3\hbar\Omega$, $k_{F0} = \pi^2 n\ell_0^2$, where n is the 3D particle density, so the period is $\lambda_0^{\mathrm{CDW}} = 1/(\pi n\ell_0^2)$. In Fig. 1 we show the transition temperature into this CDW in the LLL (dashed lines). (For repulsive interactions, we find that SDW order is dominant for all n. See Table I. At strong coupling a period of the SDW can be thought of as two separate layers of opposite spin at an effective filling factor $\nu = 1$ each.)

Our results show that CDW order always prevails for attractive contact interactions when dynamics are restricted to a single LL. However, at low rotation rates, the ground state is the BCS superconducting state (with dilute vortices). How does one reconcile these conclu-

TABLE I. With dynamics restricted to a single Landau level, the analysis of the parquet diagrams reveals a CDW instability for attractive interactions $g < 0$ and a SDW instability for $g > 0$. While the CDW is enhanced by scattering in the p-p channel, SDW order is weakened. As $n \to \infty$ the critical values, ξ_c, converge to the result for p-h ladders, $\xi_c = (2\pi)^{-1}$.

n	0	1	2	3	4	∞
$2\pi\xi_c\|_{g<0}$	0.726(4)	0.86(1)	0.91(1)	0.93(1)	0.95(1)	1
$2\pi\xi_c\|_{g>0}$	1.556(4)	1.24(1)	1.16(1)	1.13(1)	1.11(1)	1

sions? The answer lies in the coupling between LLs. Since the periods of the CDWs, λ_n^{CDW}, differ between LLs, we find that the CDW does not gain from inter-LL couplings: there are CDW instabilities at the temperatures set by our calculations for individual LLs (Table I). On the other hand, a SC state can benefit from coherence between LLs, as the Cooper pairs all have the same (zero) momentum. Thus, although SC within a single LL is less relevant than CDW, the "Josephson" coupling between LLs can stabilize a collective SC state. That said, as the topmost LL, n_{max}, depopulates, our results show that the CDW instability in this LL can occur at a higher temperature than the SC state of the entire system. In this case, the first instability (as T is reduced) is to a CDW in the Landau level n_{max}, and one expects a second instability, at lower T, to a SC state formed from the other Landau levels. (The loss of the highest LL from the SC makes little difference to its condensation energy.) In this way, we predict a supersolid ground state, involving both CDW of the topmost LL and SC order in the lower LLs. Ultimately, at sufficiently high rotation rate (or low particle density), when all particles occupy the LLL, the ground state is a CDW without superconducting order.

A striking consequence of our results is that for a rapidly rotating atomic Fermi gas, there should appear spontaneous density wave order, with a period λ_n^{CDW} that grows as the particle density in the topmost Landau level decreases. This can be a long length scale, so could be measured in experiment directly by *in situ* absorption. Clearly, the observation of the density waves requires a trap with oscillator length $\ell_{\|} > \lambda_n^{\mathrm{CDW}}$. For $\ell_{\|} < \lambda_n^{\mathrm{CDW}}$ there will be a single period of the wave, leading to a quasi-2D regime with 2D particle density in this LL equal to $n_{2d,n} = 1/(\pi\ell_0^2)$. This (incompressible) filled LL will appear as a step in the transverse density profile, as measured *in situ* or in an expansion measurement [24].

The results that we have presented are accurate far from the resonance on the BCS side, where interactions are weak. We expect the qualitative behavior to survive as the resonance is approached. While the detailed energetics of both phases cannot be relied upon for strong coupling, we find that SC is stabilized relative to CDW order for chemical potentials above the LLL as the coupling increases. Presumably, this leads to the suppression of CDW states in any but the lowest LL as one approaches the resonance. Furthermore, we note that the density wave state(s) we find on the BCS side of the resonance cannot evolve smoothly to the Bose-Einstein condensate side. A CDW of atoms, with $\nu_{\mathrm{atom}} = 2$ per period, could evolve, to retain the same period, into a CDW of tightly bound molecules with $\nu_{\mathrm{mol}} = 1/2$ per period [25]. However, there must be a phase transition separating these two states, owing to the different edge structures of the phases [26]. Thus, in contrast to the SC phase at low rotation rate, in the extreme quantum limit (at high rotation rate) tuning the interactions across the Feshbach resonance must involve a phase transition.

We thank J. Chalker, M. Gunn, D. Haldane, W. Ketterle, and D. Khmelnitskii for helpful discussions. This work was supported by the EPSRC (No. GR/S61263/01).

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$$S_{\nu\nu'} = c_X \delta(k + k') \frac{(-1)^n 2^{-N}}{\sqrt{\sqrt{2\pi} n! n'! \ell_0}} H_N(\ell_0 \sqrt{2}\bar{x}) e^{-(\ell_0 \bar{x})^2},$$

with $\bar{x} = (x - x')/2$, $N = n + n'$, and $\int dX/(2\pi)|c_X|^2 = 1$.
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