Composite Fermion Theory for Bosonic Quantum Hall States on Lattices

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We study the ground states of the Bose-Hubbard model in a uniform magnetic field, motivated by the physics of cold atomic gases on lattices at high vortex density. Mapping the bosons to composite fermions (CF) leads to the prediction of quantum Hall fluids that have no counterpart in the continuum. We construct trial states for these phases and test numerically the predictions of the CF model. We establish the existence of strongly correlated phases beyond those in the continuum limit and provide evidence for a wider scope of the composite fermion approach beyond its application to the lowest Landau level.

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Ultracold atomic gases have become a very active field of study of strongly correlated quantum systems. While dilute Bose gases are typically in a weakly interacting regime, they can be driven into regimes of strong correlations. The application of an optical lattice potential leads to a suppression of the kinetic energy relative to the interaction energy and has allowed the experimental exploration of the quantum phase transition between Mott insulator and superfluid [1]. Rapid rotation of the atomic gas also leads to a quenching of the kinetic energy into degenerate Landau levels [2], and a regime of strong interactions [3]. At a low filling factor ν (defined as the ratio of the number of particles to the number of vortices), this is predicted to lead to strongly correlated phases [4] which can be viewed as bosonic versions of fractional quantum Hall effect (FQHE) states [5]. In order to access the low filling factor regime in experiment, it may be favorable to exploit the strong interactions that are available in optical lattice systems [6,7] for which methods exist to simulate uniform rotation, or equivalently, a uniform magnetic field [8–10], and achieve high vortex density. This raises the interesting question: what are the correlated phases of atomic gases that are subjected both to a lattice and to an effective magnetic field?

In this Letter, we study the interplay between the FQHE of bosons and the strong correlation imposed by a lattice potential. At sufficiently low particle density, the effect of the lattice has been shown to have negligible impact on the nature of the continuum Laughlin state at $\nu = \frac{1}{2}$ [7,10]. We focus on the possibility that there exist strongly correlated phases which have no counterpart in the continuum, but that appear as a direct consequence of both the lattice potential and a (simulated) magnetic field. To do so, we adapt the composite fermion (CF) theory [11,12], which has been shown to accurately describe atomic Bose gases in the continuum [13,14], and apply this theory to bosons on a lattice. Within mean-field theory, the lattice leads to the intricate Hofstadter spectrum for the composite fermions [15,16]. We predict a series of incompressible phases of bosons on a lattice, characterized by special relations of the flux density n_ϕ and particle density n, and we construct trial wave functions describing these phases. From extensive exact diagonalization studies, we establish the accuracy of the composite fermion approach, notably for states for which $n = \frac{1}{2} \pm \frac{1}{2} n_\phi$; these correspond to incompressible quantum Hall states which have no counterpart in the continuum. To our knowledge, there has been no previous evidence for new FQHE states induced by a lattice potential. A previous proposal for quantum Hall states of bosons on the lattice [17] takes a different viewpoint, but remains untested.

We study a model of bosonic atoms on a two-dimensional square lattice and subjected to a uniform effective magnetic field, using the Bose-Hubbard model with Hamiltonian [8–10,18]

$$H = -J \sum_{\langle i,j \rangle} [\hat{a}_i^\dagger \hat{a}_j e^{iA_{ij}} + \text{H.c.}] + \frac{U}{2} \sum_i \hat{n}_i(\hat{n}_i - 1), \quad (1)$$

with $\hat{a}_i^{(\dagger)}$ bosonic field operators on site i, and $\hat{n}_i \equiv \hat{a}_i^\dagger \hat{a}_i$. We consider a uniform system with fixed average particle density n (per lattice site). The strength of the magnetic field is set by the flux density n_ϕ (per plaquette), defined by the condition that $\sum_\square A_{ij} = 2\pi n_\phi$. Here, $n_\phi = ma^2\Omega/(\pi\hbar)$ if this vector potential is due to rotation of the system with lattice constant a and boson mass m at the angular frequency Ω. Simulating the field by imprinting phases [8–10] directly defines $2\pi n_\phi$; such methods are likely to allow fields with $n_\phi \sim 1$. Owing to the periodicity under $n_\phi \to n_\phi + 1$, we choose $0 \leq n_\phi < 1$.

The single-particle spectrum follows from the solution of Harper's equation, and takes an intricate form, known as the Hofstadter butterfly [19]. It has a fractal structure consisting of q bands at rational flux density $n_\phi = p/q$. Signatures of this structure appear in the mean-field treatment of the Bose-Hubbard model [20,21]. We wish to determine the ground states (GS) of bosons beyond the mean-field regime, where interparticle repulsion leads to strongly correlated phases. We focus on the hard-core limit $U \gg J$, where the bosonic Hilbert space is reduced to single occupations of lattice sites $0 \leq n_i \leq 1$. In this limit, the Hamiltonian (1) can be viewed as a spin-1/2 quantum magnet. The gauge fields introduce frustration, putting this in the class of frustrated quantum spin models where unconventional spin-liquid phases can appear. Indeed, the Laughlin $\nu = \frac{1}{2}$ state studied in Ref. [10] is in the Kalmeyer-Laughlin [22] spin-liquid phase [23]. The strongly correlated phases that we describe here can be viewed as generalizations of this spin-liquid phase.

Following the application of CF theory for rotating bosons in the continuum [13], we construct composite fermions by attaching a single vortex to each boson. The CF transformation relates the flux density for the original atoms n_ϕ and the effective flux for CFs n_ϕ^* via

$$n_\phi^* = n_\phi \pm n, \tag{2}$$

where the two signs correspond to attaching vortices of opposite sign. Within a mean-field theory, the CFs are assumed to be weakly interacting, and to form a Fermi sea which fills the lowest energy states of the single-particle spectrum. Incompressible states then occur when the CFs completely fill an integer number of bands. In the continuum, the single-particle spectrum consists of Landau levels (LL), leading to an incompressible state each time an integer number, $\nu^* = n/n_\phi^*$, of CF Landau levels is filled [13,14]. Applying this logic on the lattice leads to the conclusion that the single-particle spectrum of the CFs is the Hofstadter butterfly [16], now at a flux density n_ϕ^*. Owing to the fractal structure of this spectrum, depending on n_ϕ^* there can be many such energy gaps, leading to many possible incompressible states. To determine the locations of these incompressible states, we need to know the particle densities n which completely fill an integer number of bands of the spectrum of CF's at flux n_ϕ^*. Generalizing from the continuum DOS for LLs, which is proportional to the flux density, an analysis of the lattice spectrum yields that, when filling all states up to any given gap in the Hofstadter spectrum, the relation between n and n_ϕ^* remains linear [24,25], $n = \nu^* n_\phi^* + \delta$, with an offset δ. The coefficients ν^* and δ can be determined from the Hofstadter spectrum by locating two points within the same gap. Using the reverse of the CF transformation (2), one obtains the lines of n, n_ϕ on which a nonzero gap is predicted above the CF ground state.

Within a model of noninteracting CFs the relative magnitudes of gaps follow from those in the single-particle CF spectrum. The gaps inferred under this hypothesis are shown in Fig. 1, in which the mode of flux attachment [determined by the sign in (2)] is chosen to maximize the gap. Note that the positive sign in (2) can be regarded either as negative flux attachment [26], or as attachment of the conjugate flux $1 - n$ due to the particle-hole symmetry on the lattice. Indeed, Fig. 1 shows symmetries under $n_\phi \leftrightarrow 1 - n_\phi$ and $n \leftrightarrow 1 - n$. In the hard-core limit, the Hamiltonian itself enjoys these symmetries, so the parameter space may be reduced to $0 \leq n, n_\phi \leq \frac{1}{2}$. In this quadrant, the lines emerging from the corner with $n = n_\phi = 0$ and constant filling factor $\nu \equiv n/n_\phi$ are the CF states expected in the continuum limit [13,14]. Crucially, however, Fig. 1 shows a large number of other lines. These correspond to new candidate incompressible states.

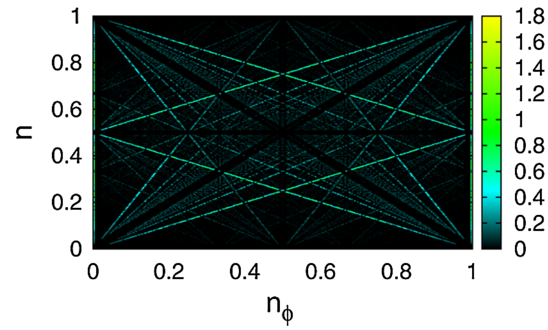

FIG. 1 (color online). Excitation gap of bosons on a lattice, with particle density n and flux density n_ϕ, as predicted by a model of noninteracting composite fermions. The bright lines show parameters (n, n_ϕ) where the model predicts the appearance of incompressible quantum fluids, and include cases (where n/n_ϕ is not constant) which are not connected to the continuum limit. We include data for $n_\phi^* = p/q$, with $q \leq 50$.

The preceding discussion conjectures candidates for new types of correlated quantum liquids of bosons on lattices. However, given that the mean-field CF theory is an uncontrolled approximation, it is important to test these predictions. There are competing condensed states on the lattice [25,27,28]. Even in the continuum limit, some of the correlated states predicted by composite fermion theory are replaced by other strongly correlated phases [5], with only $\nu = 1/2$, $2/3$, and $3/4$ appearing to be described in this form [14].

We have investigated the success of the CF construction for the Bose-Hubbard model (1) using exact diagonalization studies. We study the model for N particles on a square lattice with $N_s = L_x \times L_y$ sites, in the presence of $0 \leq N_\phi < N_s$ flux quanta. To limit finite-size effects, we impose periodic boundary conditions (pbc, discussed further below) giving the system the topology of a torus. Thus, we identify possible bulk phases and determine their properties, from which the physics of a finite system in a confining potential may be deduced within the local density approximation.

In order to compare the exact GSs with the CF theory, it is useful to have a trial CF wave function. We generalize the continuum construction [13] to allow not only for the lattice, but also for the torus geometry, for which no convenient formation exists even in the continuum limit. We construct the trial CF state for bosons in a lattice,

$$\Psi_{\text{trial}}(\mathbf{r}_1, \ldots, \mathbf{r}_N) = \Psi_{\text{J}}(\{\mathbf{r}_i\}) \times \Psi_{\text{CF}}(\{\mathbf{r}_i\}), \tag{3}$$

where Ψ_{J} and Ψ_{CF} are *fermionic* wave functions [29]. The factor Ψ_{J} effects the flux attachment (2), and represents the "Jastrow" factor of the continuum wave function [13,30].

TABLE I. Exact diagonalization results: gaps Δ and overlaps of the exact ground state with the CF trial state with negative flux attachment $\mathcal{O}_{\rm CF} = |\langle\Psi_{\rm trial}|\Psi_{\rm ex}\rangle|^2$. We also give the Hilbert-space dimension dim($\mathcal{H}$) for hard-core bosons. Where negative gaps are indicated, the CF state is the first excited state.

n	n_ϕ	N	L_x	L_y	Δ	$\mathcal{O}_{\rm CF}$	dim($\mathcal{H}$)
1/7	3/7	2	2	7	0.156	0.437	91
1/7	3/7	3	3	7	0.156	0.745	1330
1/7	3/7	4	4	7	−0.032	0.2753[a]	20.5 k
1/7	3/7	5	5	7	0.0401	0.5631	324 k
1/7	3/7	6	6	7	0.0455	0.3284	5.2 M
1/9	4/9	2	2	9	0.113	0.3603	153
1/9	4/9	3	3	9	0.241	0.8407	2925
1/9	4/9	4	4	9	−0.036	0.1515[a]	58.9 k
1/9	4/9	4	6	6	0.071	0.3061	58.9 k
1/9	4/9	5	5	9	0.0945	0.4585	1.2 M
1/9	4/9	6	6	9	−0.0154	0.1957[a]	25.8 M

[a]Overlap shown for first excited state.

However, we define this factor in a form suitable on a lattice and in the torus geometry, instead of using the continuum form of $\Psi_{\rm J}$ [30]. We note that $\Psi_{\rm J}$ corresponds to a filled Landau level of fermions at flux $\mp N$. We generate $\Psi_{\rm J}$ on the lattice as the Slater determinant $\Psi_{\rm J} = \det[\Phi_\alpha^{\mp N}(\boldsymbol{r}_\beta)]$, describing N fermions occupying the N lowest energy states on the lattice with $\mp N$ flux indexed by $\alpha = 1, \ldots, N$. This flux attachment is different from that described by Chern-Simons theories on the lattice [31], as it also includes an amplitude-modulation of the wave function. The factor $\Psi_{\rm CF}$ is the wave function of the CFs in the resulting effective field (2). This is the Slater determinant $\Psi_{\rm CF} = \det[\Phi_\alpha^{N_\phi^*}(\boldsymbol{r}_\beta)]$ of the N lowest single-particle CF states at flux $N_\phi^* = N_\phi \pm N$. For the cases derived above (and illustrated in Fig. 1), these numbers N and N_ϕ^* are such that the CFs fill an integer number of bands. Note that, in contrast to the continuum limit where the GSs have been studied within the lowest LL limit [5], our CF state (3) does *not* include a projection to the lowest LL. This is appropriate for the hard-core model that we study, since (3) vanishes when the positions of any two bosons coincide.

The description of the trial state (3) is completed by discussing the pbc imposed on each of the functions. In the most general case, one introduces twisted boundary conditions for the bosons [7,32], defined by the phases $\theta_\mu = (\theta_x, \theta_y)$, such that magnetic translations of a boson around the two cycles of the torus (by L_μ in the μ-direction) act as $\Psi \to \exp[i\theta_\mu]\Psi$. In the trial state (3), one may choose boundary phases for the Jastrow- and CF-parts independently, defining $\theta_\mu^{\rm J}$, $\theta_\mu^{\rm CF}$ (affecting only the single-particle states Φ_α entering the Slater determinants). The sum of these phases is constrained to match the pbc for the bosons $\theta_\mu^{\rm J} + \theta_\mu^{\rm CF} = \theta_\mu$. This leaves the freedom to vary $\theta_\mu^{\rm J} - \theta_\mu^{\rm CF}$, which is a crucial ingredient to our construction: it allows one to generate the set of states responsible for the nontrivial GS degeneracy of these topologically ordered phases on a torus [33]. It is easy to show that, with this freedom, in the continuum limit the wave functions (3) reproduce the two continuum Laughlin states at $\nu = 1/2$ [32].

As an initial test, we have computed the overlaps $|\langle\Psi_{\rm trial}|\Psi_{\rm exact}\rangle|^2$ of our trial wave functions (3) with the GSs on the lattice at $\nu = \frac{1}{2}$ as a function of n_ϕ. The overlaps (not shown) are very close to those found with the continuum Laughlin wave functions (closely reproducing Fig. 2 of [10]). Thus, the continuum [32] and lattice states (3) are very similar, up to the flux density $n_\phi \simeq 0.3$ at which both fail to describe the exact GS.

Using our general construction (3), we can study for the first time the influence of the lattice structure on other continuum CF states. The state at $\nu = \frac{2}{3}$ has a GS degeneracy $d_{\rm GS} = 3$ [32]. For each of the three lowest states of the exact spectrum $|\Psi_{\rm ex}^{(i)}\rangle$, we find the trial CF state with maximal overlap $|\Psi_{\rm trial}^{(i)}\rangle$ and give their average overlap in Fig. 2. The overlap is high and drops only above flux densities of $n_\phi \simeq 0.35$. Previous numerical evidence for this CF state is restricted to the lowest Landau level [5]. Our results show that, for sufficiently small n_ϕ, the CF state (3) also describes the GS for hard-core interactions (where LL mixing is strong).

Let us now return to the main focus of this Letter: the new CF states that appear on the lattice. To investigate these states numerically, we focus on the CF series derived from the most dominant gap in a subcell of the Hofstadter spectrum (cell L_1 [19]), leading to a sequence with $n_\phi = \frac{1}{2} - \frac{1}{2}n$. To be able to study several different system sizes for some states in this class, we select two points where (n, n_ϕ) are fractions with small denominators, and the density n is low enough to avoid competition with the continuum Laughlin state: $(\frac{1}{7}, \frac{3}{7})$, and $(n, n_\phi) = (\frac{1}{9}, \frac{4}{9})$.

We find multiple pieces of evidence for the formation of strongly correlated incompressible phases at these values of (n, n_ϕ). First, an analysis of the eigenvalues of the single-particle density matrix of the GS shows that, as the system size N increases, there are N eigenvalues of order one. Thus, there is no evidence for condensation (an eigenvalue that grows with N), so the GS is likely uncondensed and strongly correlated. Second, the spectra at these

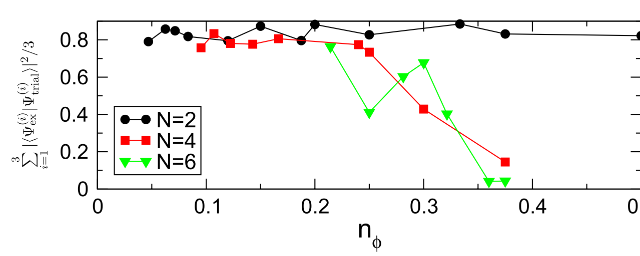

FIG. 2 (color online). Average overlap of the CF states with the exact eigenstates in the (approximately) threefold degenerate GS manifold of the $\nu = \frac{2}{3}$ state.

densities typically show a single GS separated by a large gap (see Table I). The gaps we find are larger than the typical spacing of higher excited states, or the gaps at typical spectra at nearby flux densities [34]. This indicates that the system may be an incompressible liquid with a nondegenerate GS on the torus. This is consistent with the CF state, in which one expects a GS degeneracy of one, applying the reasoning of Ref. [16].

Further direct evidence for the CF phase is obtained by taking the overlap of the exact GSs with the trial CF states (3). As detailed in Table I, we find that, in general, the trial CF states have significant overlap with the exact GS. Notable exceptions occur for certain cases ($N = 4$ for $n = 1/7$ and $n = 1/9$, and $N = 6$ for $n = 1/9$) where the exact GS has a different momentum from the CF state so the overlap vanishes identically. In these cases, we find large overlap of the CF state with the lowest lying excited state (as shown in Table I). We account for this behavior as arising from the existence of a competing broken-symmetry "stripe" phase that is stabilized by delocalization of the particles around the short direction, similar to finite-size effects in continuum studies on the torus [35]. This interpretation is confirmed by our studies at $n = 1/9$, which show that the GS is sensitive to the lattice geometry (two aspect ratios $L_x \times L_y = 4 \times 9$ and 6×6 are available for $N = 4$ at $n = 1/9$). The GS reverts to be of the CF form for the more isotropic aspect ratio. Unfortunately, no geometry with smaller aspect ratio is available for the systems ($N = 4$ at $n = 1/7$ and $N = 6$ at $n = 1/9$). Still, our results indicate that, for the system at $(n, n_\phi) = (\frac{1}{7}, \frac{3}{7})$, the composite fermion state dominates the competing (striped) state at large system sizes, and maintains a very high overlap with the exact GS. A similar trend is evident for $(n, n_\phi) = (\frac{1}{9}, \frac{4}{9})$, but here the available geometries at $N = 6$ are still very asymmetric, so we cannot confirm the preference of the CF state in this case.

While a large overlap with the trial CF state is highly suggestive that the phase is of the CF type, it is very useful to have other tests of the *qualitative* features of the state. As noted above, the nondegenerate GS is consistent with the expected topological degeneracy of the CF state. Another important qualitative test is provided by Chern numbers [7,36], which provide a highly nontrivial test of the existence and nature of the topological order of a many-body quantum phase. We have evaluated the Chern number $\mathcal{C}$ for the GSs with nonzero overlap with the CF states for systems up to $N = 5$. In all cases, we find that $\mathcal{C} = 2$. This is the value expected for the CF phase [16]. This agreement lends very strong evidence that the phase appearing in the numerics is of the form predicted by the CF theory.

In conclusion, we have presented numerical evidence for novel types of correlated quantum fluids of bosons on lattices; these are FQHE states which have no counterpart in the continuum limit.

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