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PAPER

Synthetic gauge fields for lattices with multi-orbital unit cells: routes towards a π -flux dice lattice with flat bands

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Gunnar Möller¹ and Nigel R Cooper² ¹ Functional Materials Group, School of Physical Sciences, University of Kent, Canterbury CT2 7NZ, United Kingdom² TCM Group, Cavendish Laboratory, University of Cambridge, Cambridge CB3 0HE, United KingdomE-mail: G.Moller@kent.ac.uk**Keywords:** optical flux lattice, synthetic gauge fields, flat bands, Berry phases, optically dressed states, changing lattice topology, designing optical lattices in reciprocal space**Abstract**

We propose a general strategy for generating synthetic magnetic fields in complex lattices with non-trivial connectivity based on light–matter coupling in cold atomic gases. Our approach starts from an underlying optical flux lattice in which a synthetic magnetic field is generated by coupling several internal states. Starting from a high-symmetry optical flux lattice, we superpose a scalar potential with a super- or sublattice period in order to eliminate links between the original lattice sites. As an alternative to changing connectivity, the approach can also be used to create or remove lattice sites from the underlying parent lattice. To demonstrate our concept, we consider the dice lattice geometry as an explicit example, and construct a dice lattice with a flux density of half a flux quantum per plaquette, providing a pathway to flat bands with a large band gap. While the intuition for our proposal stems from the analysis of deep optical lattices, we demonstrate that the approach is robust even for shallow optical flux lattices far from the tight-binding limit. We also provide an alternative experimental proposal to realise a synthetic gauge field in a fully frustrated dice lattice based on laser-induced hoppings along individual bonds of the lattice, again involving a superlattice potential. In this approach, atoms with a long-lived excited state are trapped using an ‘anti-magic’ wavelength of light, allowing the desired complex hopping elements to be induced in a specific laser coupling scheme for the dice lattice geometry. We conclude by comparing the complexity of these alternative approaches, and advocate that complex optical flux lattices provide the more elegant and easily generalisable strategy.

1. Introduction

The creation of synthetic gauge fields in cold atomic gases provides new opportunities for realising exotic emergent quantum phases [1–5]. Prominent target phases include vortex lattices [6] and, at high flux density, bosonic counterparts of the continuum fractional quantum Hall states [7, 8]. When both a (synthetic) field and a lattice potential are present, the continuum quantum Hall states are predicted to persist for appreciable flux densities n_ϕ per plaquette [9]. In addition, new classes of quantum Hall states, stabilized only due to the presence of a periodic lattice potential, emerge at larger values of n_ϕ owing to the underlying structure the Hofstadter spectrum [10–14], and in particular owing to the presence of single-particle bands with higher Chern numbers $|C| > 1$ [14, 15].

Early experiments on synthetic gauge fields relied on using rotation to emulate magnetic fields [1, 16]. However, in this approach it is exceedingly difficult experimentally to avoid heating due to asymmetric trapping potentials, so the strongly interacting regime of low density in the lowest Landau level remains out of reach. Prompted in part by the exciting outlook for the creation of new phases of matter, there has been much progress with new theoretical proposals and the experimental realizations for schemes of simulating artificial gauge fields [17–25]. Further impetus for synthetic fields stems from the prospect of realising topological flat bands in

condensed matter systems—where spin–orbit coupling may provide suitable complex hopping elements in a tight-binding representation—sharpening the focus on the underlying commonality of flat single-particle bands with non-zero Chern number [26–32], and more detailed characteristics of their band geometry [33–36]. Currently no clear target systems realising synthetic magnetic flux have been identified in the solid state, while cold atoms provide a range of successful realizations³. Early achievements include the square lattice with staggered magnetic flux [38, 39] that was generated by suitably tailored laser-induced hoppings [17, 22]. More recently, experiments have achieved homogeneous magnetic flux using related approaches [40–43]. The Chern bands of the Haldane model [27] were also successfully engineered using a lattice shaking approach [44]. Features of the non-trivial band single band topology have been successfully identified [45, 46]. Another groundbreaking line of research has exploited spatially dependent dressed states of atoms in order to create a Berry phase emulating the Aharonov–Bohm effect of charged particles moving in a magnetic field [21]. The experimental realization of this approach [21] has prompted further theoretical developments in order to maximize the achievable flux density in so-called optical flux lattices [24, 47]. These systems rely on modulating the optical dressed states of multi-state atoms on the scale of the optical wavelength, thus accessing the smallest possible length scales for light–matter coupled systems, and provide a viable route to observe fractional quantum Hall physics [2, 48, 49]. Experimental progress has been reported on the intimately related case of emulating spin–orbit coupling in two-dimensional systems [50–52].

So far, attempts to emulate optical lattices with synthetic gauge fields have focused on continuum gases or on simple optical lattice geometries such as square and triangular lattices [53]. However, optical lattices without gauge fields have already been demonstrated for more complex geometries such as the kagome lattice [54], which is achieved by removing *sites* from an underlying triangular lattice. Lattice geometry plays a particularly important role in the presence of magnetic flux, as it can affect the single-particle spectrum dramatically. Indeed, the elegant Hofstadter butterfly seen in the spectrum of the square lattice [55] is strongly altered in other geometries such as the triangular [56] or hexagonal lattices [57]. This provides a strong incentive to achieve synthetic gauge fields in a number of different lattice geometries.

It is well understood how complex lattice geometries can be realised in scalar optical lattices by exploiting the superposition of several optical lattice potentials [54, 58, 59]. In this paper, we explore how this design principle can be extended to create optical flux lattices with non-trivial connectivity by superposing scalar sub-/superlattice potentials to an optical flux lattice that generates non-trivial Berry phases from adiabatic motion within the space of internal states of the trapped atoms. We demonstrate that a scalar potential may be used to either remove *bonds* or *sites* from an underlying optical flux lattice of simpler geometry, as well as to split individual sites into multiple wells, all the while keeping the synthetic field intact. The basic principle for controlling bonds can be understood from a tight-binding picture: the dynamics of atoms in an optical lattice arises from hopping processes between local Wannier states that are localized in the minima or wells of the optical potential [60]. The amplitude of hopping processes is given by the overlap of these wave functions. As the overlap is dominated by the exponential tails penetrating the potential maxima that separate adjacent wells, hopping is extremely sensitive to the magnitude of this potential. Therefore, hopping can be almost completely suppressed by increasing the height of the potential maximum between two wells when a scalar potential is added at those locations. Generally, we wish to suppress bonds on a periodic sublattice of an underlying optical flux lattice, so this can be implemented by superposing an additional scalar optical lattice potential which acts equally on all internal states. In practice, optical flux lattices operate in an intermediate coupling regime where the lattice potential is sufficiently shallow for atoms to occupy any position in space. One of the main results of the current work is to demonstrate that complex optical flux lattices can operate in a regime of weak coupling that remains far from the tight-binding limit: we provide a specific example showing that the dispersion of the tight-binding picture is reproduced closely even in the regime of shallow lattice depth with potential depth of order of the atomic recoil energy.

In order to demonstrate our general principle, we propose and analyse in detail a new realization for synthetic fields in the dice lattice (also known as \mathcal{T}_3 -lattice) where the specific flux density of $\Phi = \Phi_0/2$ per plaquette yields a particularly surprising band structure with three pairs of perfectly flat bands that conserve time-reversal symmetry [61]. The flat bands and compactly localized single-particle states found in this lattice are caused by a phenomenon of destructive interference known as Aharonov–Bohm caging [61]. This regime would be particularly well suited to reach interesting correlation phenomena [62–64], but previous proposals for synthetic fields in a dice lattice geometry have focused on a different regime with dispersive Chern bands [65]. Unlike most flat band models achievable in cold atoms [66], the flat bands of the π -flux dice lattice model are fully gapped. Owing to the flatness of the band dispersion, even weak interactions give rise to exotic phases in the dice lattice model, including a superfluid phase in the half filled lowest band [63] as well as highly degenerate

³ We also note the successful observation of fractional Chern insulating phases in graphene-based heterostructures under strong physical magnetic fields [37].

vortex lattice configurations at larger density [62, 63] that provide a playing field for order-by-disorder phenomena [64]. Hence, akin to the physics of flat band ferromagnetism [67, 68], the dominant phases in the dice lattice provide interesting alternatives to more conventional features of Bose condensation in dispersive bands [39].

To further contrast the new proposal with more conventional techniques, we also present an alternative design for a dice lattice with a synthetic π -flux based on alkaline earth atoms trapped by light near their anti-magic wavelength. We describe a set-up creating laser-induced hoppings according to the connectivity of the dice lattice, that can be realised using far-detuned transitions following [22]. Our design explicitly constructs the tight-binding Hamiltonian within the magnetic unit cell, containing a total of six sites, which is repeated due to the inherent periodicity of the trapping lasers. We find that the two designs involve a similar number of laser sources, and we argue that requirements on phase stability favour the optical flux lattice approach.

The paper is organised as follows. In section 2, we review how the concept of adiabatic motion in optical dressed states enables the creation of optical flux lattices, and we establish our notations. In section 3, we introduce the idea of changing lattice connectivity by removing bonds from an optical flux lattice at the level of a tight-binding approach, and perform an analysis of its translational symmetries. In section 4 we detail how the idea can be exploited to realize the dice lattice geometry with half a flux quantum per plaquette, focusing on a tight-binding picture. Section 5 gives the general formalism for studying optical flux lattices beyond the tight-binding limit in reciprocal space, and we use the example of the dice lattice geometry to demonstrate the role of spin-translation symmetries of the flux lattice Hamiltonian. In section 6, we provide detailed calculations of the band structure for realistic parameters in our dice flux lattice geometry, focusing on the limit of a shallow lattice. Section 7 provides the alternative design, based on laser-induced hoppings in a deep optical lattice, and we conclude in section 8.

2. Background: optical flux lattices

The optical flux lattice approach is motivated by the principle of adiabatic motion of atoms, such that they remain in their local ground-state $|\Psi(\mathbf{r})\rangle$ along their trajectory $\mathbf{r}(t)$ [24], taken here to be confined within the two-dimensional x - y -plane. Upon completion of a closed path \mathcal{C} , the wavefunction of the atoms acquires a geometrical Berry phase $\gamma = \oint_{\mathcal{C}} q\mathbf{A} \cdot d\mathbf{l}$, given by the line integral over the (real space) Berry connection $q\mathbf{A} = i\hbar\langle\Psi|\nabla\Psi\rangle$ (with a fictitious charge q) [21]. This geometric phase mimics the Aharonov–Bohm coupling of a charged particle to the vector potential of a physical magnetic field, which has the same form. It also useful to think of the corresponding flux density $n_\phi = q/h(\nabla \times \mathbf{A}) \cdot \hat{\mathbf{e}}_3$.

The presence of vortices in the Berry connection allows one to achieve flux densities of order one magnetic flux quantum per unit cell of the optical flux lattice. Here, we will consider as our starting point the explicit example of the triangular flux lattice of [24] for a two-state system with the Hamiltonian

$$\hat{\mathcal{H}}_\Delta = \frac{\hat{\mathbf{p}}^2}{2m}\mathbf{1} + \mathcal{V}\mathbf{M}(\mathbf{r}) \cdot \hat{\boldsymbol{\sigma}}, \quad (1)$$

where $\mathbf{1}$ is the 2×2 identity matrix in spin-space, $\hat{\boldsymbol{\sigma}} = (\hat{\sigma}_1, \hat{\sigma}_2, \hat{\sigma}_3)$ is the vector of Pauli matrices, and \mathcal{V} is the depth of the optical lattice. We consider the triangular optical lattice potential described by

$$\mathbf{M}(\mathbf{r}) = \cos(\vec{\kappa}_1 \cdot \mathbf{r})\hat{\mathbf{e}}_1 + \cos(\vec{\kappa}_2 \cdot \mathbf{r})\hat{\mathbf{e}}_2 + \cos(\vec{\kappa}_3 \cdot \mathbf{r})\hat{\mathbf{e}}_3, \quad (2)$$

where $\hat{\mathbf{e}}_i$ are the cartesian unit vectors, and the wave vectors $\vec{\kappa}_1 = (1, 0)\kappa$, $\vec{\kappa}_2 = (1/2, \sqrt{3}/2)\kappa$, and $\vec{\kappa}_3 = (-1/2, \sqrt{3}/2)\kappa$ are chosen to yield a lattice potential with minima separated by a lattice vector a , i.e., we require $\kappa = \frac{2\pi}{\sqrt{3}a}$. In our notations, we highlight constant vectors defined by externally imposed geometrical features such as $\vec{\kappa}_i$ in bold-face with an additional arrow, while vectors representing variables like \mathbf{r} are denoted in simple bold font. Note that specific implementations of a triangular optical flux lattice such as (1) may be realised by various optical coupling schemes. Detailed implementations have been presented elsewhere (see, e.g., [47]), so we shall work with the simplest model in the current paper.

In the adiabatic limit $m \rightarrow \infty$, it is easily checked that the Hamiltonian (1) has eigenvalues $E_\pm(\mathbf{r}) = \pm\mathcal{V}|\mathbf{M}|$, and the local Bloch vector for the lower band, $\hat{\mathbf{n}} = \langle\Psi_-(\mathbf{r})|\hat{\boldsymbol{\sigma}}|\Psi_-(\mathbf{r})\rangle$, is simply given by the direction of $-\mathbf{M}$, i.e., $\hat{\mathbf{n}} = -\hat{\mathbf{M}} \equiv -\mathbf{M}/|\mathbf{M}|$. The states $|\Psi_\pm\rangle$ are also the eigenstates for the class of Hamiltonians $\hat{\mathcal{H}}' = \hat{\mathcal{H}}_\Delta + \hat{\mathcal{V}}_s$, for arbitrary scalar (i.e., spin-independent) potentials $\hat{\mathcal{V}}_s(\mathbf{r}) = V_s(\mathbf{r})\hat{\mathbf{1}}$. The energy landscape for the unperturbed triangular flux lattice (1) is shown in figure 1. Note that the unit cell of this lattice encloses two flux quanta within an area containing four local minima of the energy, which we can think of as four lattice sites in the tight-binding limit of a deep optical flux lattice [24]. For our choice of units, the lattice vectors spanning the unit cell are given by $\vec{a}_1 = (\sqrt{3}, -1)a$, and $\vec{a}_2 = (0, 2)a$, as highlighted in figure 1.

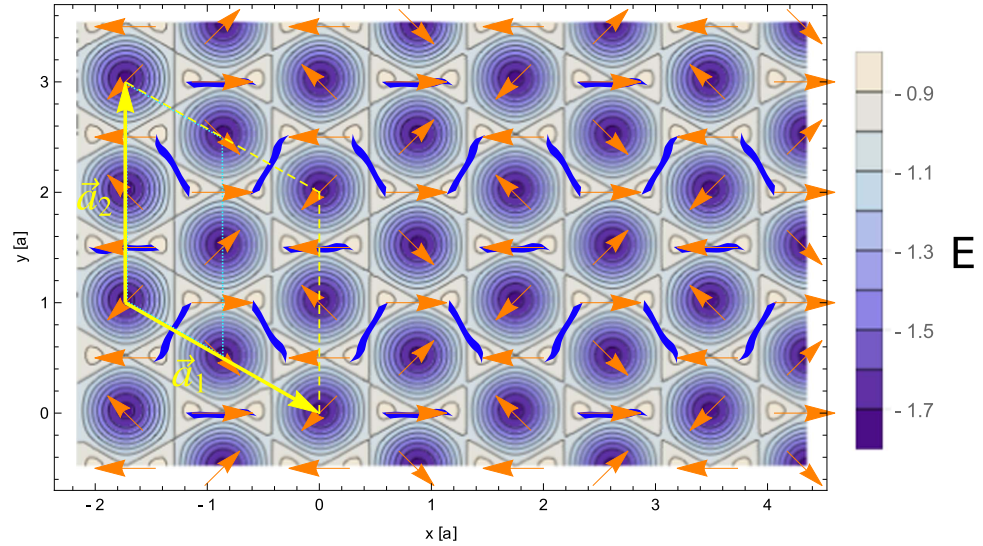


Figure 1. Contour plot of the energy landscape for the triangular optical flux lattice with two flux quanta per unit cell of [24], the starting point for our construction. Orange arrows show the in-plane components of the local Bloch vector. The unit cell is spanned by the vectors \vec{a}_1, \vec{a}_2 , contains 4 triangular lattice sites, and encloses 2 flux quanta. Thanks to a spin-translation symmetry, this can be reduced to a reduced unit cell of size $[\vec{a}_1/2, \vec{a}_2]$ (dotted cyan lines). In this paper, we show how this flux lattice can be modified to yield an optical dice flux lattice by eliminating bonds: a dice lattice is obtained by impeding tunnelling across the links which are crossed out with blue wavy lines.

The periodicity of the energy landscape suggests that the Hamiltonian (1) has a higher translational symmetry than that by the above-mentioned lattice vectors \vec{a}_i . While energetically equivalent, the eigenstates at the four energy minima in the unit cell are distinct. However, the higher symmetry of the Hamiltonian can be revealed by generalized translation operators that incorporate a rotation in spin-space [24]. Available spin-translation operators are

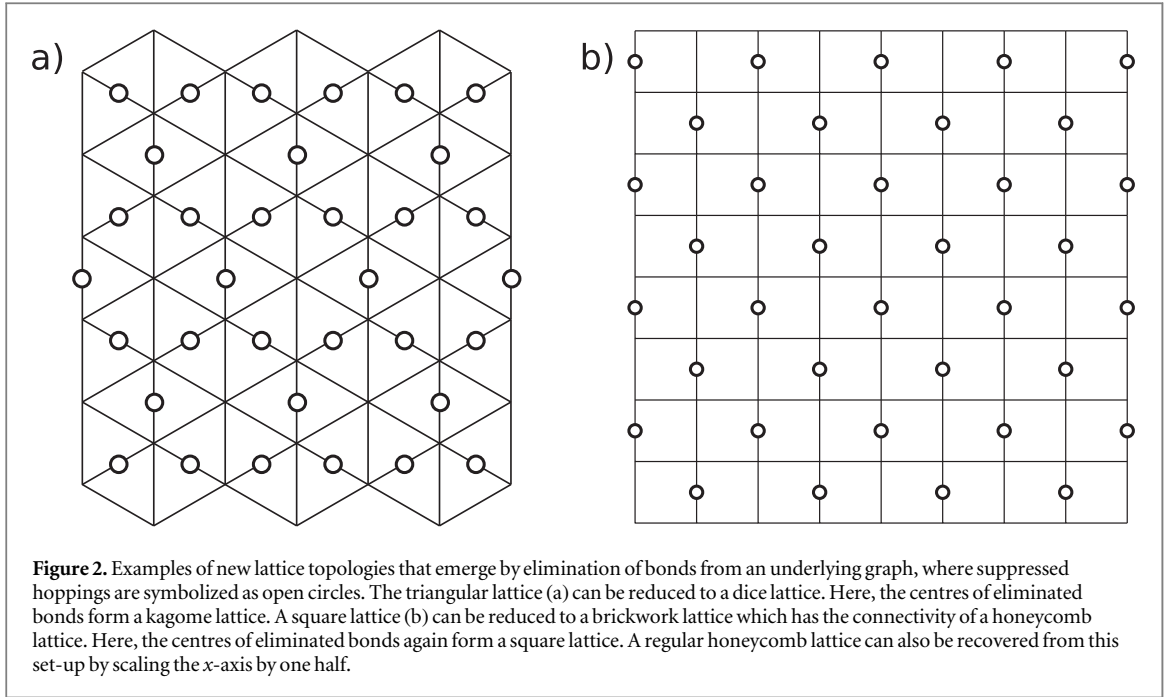
$$\hat{T}_1 = \hat{\sigma}_2 e^{\frac{1}{2} \vec{a}_1 \cdot \nabla}, \quad \hat{T}_2 = \hat{\sigma}_1 e^{\frac{1}{2} \vec{a}_2 \cdot \nabla}, \quad (3)$$

with $[\hat{T}_i, \hat{\mathcal{H}}_\Delta] = 0$ ($i = 1, 2$), but $[\hat{T}_1, \hat{T}_2] \neq 0$. Nonetheless, we find that $[\hat{T}_1, \hat{T}_2^2] = 0$, so we can classify the eigenvalues of $\hat{\mathcal{H}}_\Delta$ with the quantum numbers of both \hat{T}_1 , and $\hat{T}_2^2 = \exp(\vec{a}_2 \cdot \nabla) \equiv \hat{K}(\vec{a}_2)$, as the latter reduces to a regular translation $\hat{K}(\vec{a}_2)$ by \vec{a}_2 . For a detailed discussion of these symmetry operations in the triangular lattice, see [24].

3. Changing lattice topology via scalar potentials

In the deep optical lattice limit, we can consider optical flux lattices as tight-binding models where motion between two ‘sites’ or local minima of the energy landscape is described by a tight-binding model with complex hopping elements. We now examine how a change in the lattice topology emulated by optical flux lattices is achieved either by ‘removing sites’ or by ‘removing bonds’ in this tight-binding model, as was already achieved for scalar optical lattices [58, 54]. As we will demonstrate below, this idea can indeed also be realised in optical flux lattices by applying an additional scalar optical lattice potential to either suppress lattice sites or the connectivity between them, while the distribution of flux generated by the underlying optical flux lattice is kept intact.

Some examples of cutting bonds are visualized in figure 2. There are already similar experimental realisations of tuneable optical lattices obtained by superposing multiple standing waves [58, 54]. An additional consideration for flux lattices arises in the tight-binding limit, where flux through each plaquette is defined only modulo 2π . As the elimination of links merges the two adjoining plaquettes into a single one, this construction yields non-trivial flux lattices only if the total flux in the resulting merged plaquette is not an integer multiple of the flux quantum Φ_0 . Similarly, the removal of sites also merges several adjoining plaquettes, so the same consideration applies. For example, a hexagonal lattice can be obtained by removing a sublattice of sites of an underlying triangular lattice. In this case, six neighbouring triangular plaquettes are joined into a hexagonal one, so this yields non-trivial results if the flux per triangular plaquette is not a multiple of $\Phi_0/6$.



4. Case study: the dice lattice

For the remainder of this paper, we focus on a case study of eliminating bonds in a triangular flux lattice. Alongside the elementary unit cell of the flux lattice, figure 1 highlights the bonds that need to be severed in the triangular lattice so as to reduce its connectivity to a dice lattice geometry. As shown more clearly in figure 2(a), we find that mid-points of these bonds form a kagome lattice with lattice constant $a' = \sqrt{3}/2a$. From figure 1, it is also clear that the pattern of eliminated bonds has a different periodicity as the unit cell $[\vec{a}_1, \vec{a}_2]$ of the triangular optical flux lattice. This will be further discussed, below.

In our cold atom realization of an optical dice flux lattice, the maxima of an additional scalar optical potential are aligned with the centre points of the bonds of an underlying triangular optical flux lattice. As experiments by the Stamper-Kurn group demonstrate, an attractive kagome optical lattice can be achieved by combining a blue-detuned (i.e., regions of high intensity are repulsive) short-wavelength triangular optical lattice with a red-detuned (attractive) triangular lattice of twofold lattice constant [54]. Experimentally, it is difficult to keep these two lattices in register, but this challenge has been successfully addressed [54]. Here, we require a *repulsive* kagome lattice, which is rotated by $\pi/6$ with respect to an underlying triangular optical flux lattice (1), again implying that the two light potentials have to be kept in phase as in the kagome lattice realisation of [54]. The corresponding optical potential is formed by a red-detuned short-wavelength scalar optical lattice V_{SW} at wave number $\kappa^\perp = 2\kappa/\sqrt{3}$, as well as a blue-detuned long-wavelength scalar superlattice V_{LW} with wave number $\kappa^\perp/2$. The full Hamiltonian of our optical dice flux lattice is then obtained by superposing all three components

$$\hat{\mathcal{H}}_{\text{dice}}(r, b) = \hat{\mathcal{H}}_\Delta + [rV_{\text{SW}}(\hat{\mathbf{r}}) + bV_{\text{LW}}(\hat{\mathbf{r}})]\mathbf{1}. \quad (4)$$

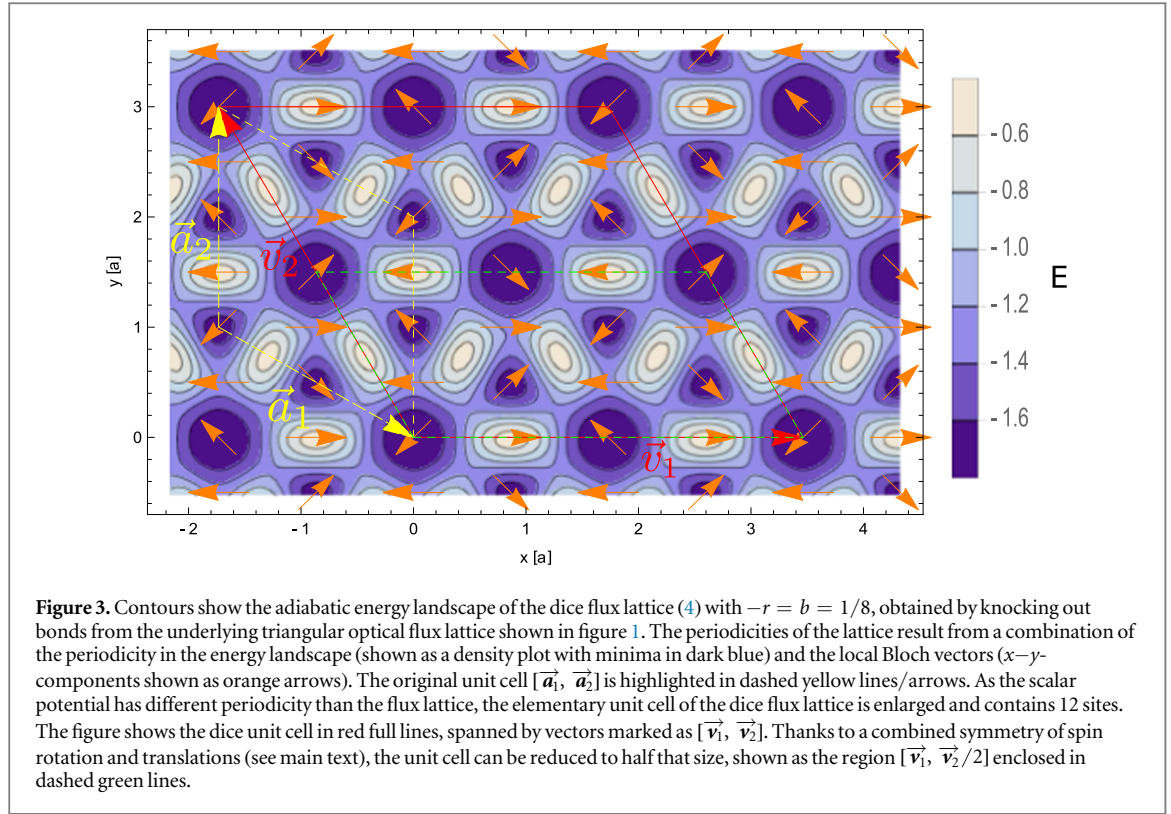
Here, the parameters $b > 0$ and $r < 0$ give the amplitude of the scalar beams relative to the spin-dependent fields, and the explicit form of the required short- and long-wavelength potentials are given by

$$V_{\text{SW}}(\mathbf{r}) = \mathcal{V}[\sin^2(\vec{\kappa}_1^\perp \cdot \mathbf{r}) + \sin^2(\vec{\kappa}_2^\perp \cdot \mathbf{r}) + \sin^2(\vec{\kappa}_3^\perp \cdot \mathbf{r})] \quad (5)$$

for the red-detuned beam that is attractive, and that should thus contribute with an amplitude $r < 0$, and

$$V_{\text{LW}}(\mathbf{r}) = \mathcal{V}\left[\sin^2\left(\frac{\vec{\kappa}_1^\perp}{2} \cdot \mathbf{r}\right) + \sin^2\left(\frac{\vec{\kappa}_2^\perp}{2} \cdot \mathbf{r}\right) + \sin^2\left(\frac{\vec{\kappa}_3^\perp}{2} \cdot \mathbf{r}\right)\right] \quad (6)$$

for the blue-detuned beam that should provide a repulsive potential with an amplitude $b > 0$, and $\vec{\kappa}_i^\perp = 2/\sqrt{3}\hat{e}_3 \wedge \vec{\kappa}_i$ throughout. Note that both these contributions are scalar, i.e., they are diagonal in spin-space. In the adiabatic limit (i.e., disregarding kinetic energy), the local energy eigenvalues are readily obtained as $E_\pm^{\text{dice}}(r, b) = \pm\mathcal{V}|\mathbf{M}| + rV_{\text{SW}}(\mathbf{r}) + bV_{\text{LW}}(\mathbf{r})$, and the local eigenstates are unchanged with respect to the triangular optical flux lattice.



Let us now discuss the symmetries of the optical dice flux lattice Hamiltonian. As we noted previously, it does not have the full translational symmetry of the triangular optical flux lattice. The resulting situation is best discussed in terms of figure 3, which shows the energy landscape (contours; darker blue indicates minima), as well as the x - y -components of the local Bloch vector (orange arrows). In the presence of the scalar potentials (5), (6), the energy landscape contains lattice sites with three different profiles: the most prominent minima form the ‘hubs’ or sixfold connected sites of the dice lattice, such as the one at the origin $\mathbf{r} = (0, 0)$. They are surrounded by six smaller minima, the ‘rims’ or threefold connected sites. These are slightly triangular and can be either pointing upwards [such as at $\mathbf{r} = (\sqrt{3}/2, 1/2)a$] or downwards [as at $\mathbf{r} = (0, 1)a$]. In addition, lattice sites differ in terms of the spin-content of the local wavefunction. Looking at the in-plane components of the local Bloch vectors, it is apparent that a fundamental unit cell of our optical dice flux lattice is enclosed by the vectors marked in figure 3 as $\vec{v}_1 = (2\sqrt{3}, 0)a$, and $\vec{v}_2 = (-\sqrt{3}, 3)a$, which connect hubs with identical Bloch vectors. Due to the distinct periodicities, this unit cell contains 12 sites of the underlying triangular lattice so it is enlarged threefold with respect to the unit cell of the original triangular optical flux lattice.

The Hamiltonian (4) contains an additional symmetry, which can be constructed in terms of the spin-translation operators $\hat{T}_{1,2}$ in (3). Let us construct suitable spin-translations $\hat{S}_{1,2}$ along the half lattice vectors $\frac{1}{2}\vec{v}_{1,2}$. These can be expressed in terms of $\hat{T}_{1,2}$ as:

$$\hat{S}_1 = \hat{T}_1^2 \hat{T}_2 = \hat{\sigma}_1 e^{\frac{1}{2}(2\vec{a}_1 + \vec{a}_2) \cdot \nabla}, \quad (7)$$

$$\hat{S}_2 = \hat{T}_1^{-1} \hat{T}_2 = i\hat{\sigma}_3 e^{\frac{1}{2}(-\vec{a}_1 + \vec{a}_2) \cdot \nabla}. \quad (8)$$

We note that both \hat{S}_i commute with the Hamiltonian, i.e. $[\hat{S}_{1,2}, \hat{\mathcal{H}}_{\text{dice}}] = 0$. Furthermore, their squares are simple translations, which confirms that we have chosen the unit cell correctly. For instance, $\hat{S}_1^2 = \hat{\sigma}_1^2 e^{(2\vec{a}_1 + \vec{a}_2) \cdot \nabla} = e^{\vec{v}_1 \cdot \nabla}$, which equals a pure translation $\hat{K}(\vec{v}_1)$ under the lattice vector \vec{v}_1 . However, the translations \hat{S}_1 and \hat{S}_2 do not commute with each other, as $[\hat{S}_1, \hat{S}_2] \neq 0$. Given that \hat{S}_2 is diagonal in spin-space, we select this operator as our supplementary symmetry in formulating the single-particle Hilbert-space, and we can then use the eigenvalues of the set of commuting operators $\hat{\mathcal{H}}, \hat{S}_1^2$, and \hat{S}_2 to label eigenstates. This results in a reduced unit cell in real space, spanned by $[\vec{v}_1, \vec{v}_2/2]$, as shown in green dashed lines in figure 3, such that eigenstates in the remainder of the full unit cell can be recovered by applying \hat{S}_2 to their symmetry related points in the reduced cell.

5. Spin-translation symmetry in shallow flux lattices

The arguments of the preceding section can be placed on a more robust foundation by considering the full Hamiltonian of the flux lattice beyond the tight-binding limit, i.e., including kinetic energy. In order to capture the effect of the kinetic energy term, it is convenient to study the flux lattice Hamiltonian as a tight-binding model in reciprocal space [69]. Here, we review and extend this formalism to take into account the spin-translation symmetries, as realised by the operators \hat{S}_1, \hat{S}_2 identified above.

5.1. General formalism

Having identified the periodicity of the problem in the real-space unit cell spanned by $[\vec{v}_1, \vec{v}_2]$, we know that the wave functions in reciprocal space are defined on a fundamental Brillouin zone (BZ) spanned by the reciprocal lattice vectors

$$\vec{g}_i = \epsilon_{ij} \frac{2\pi \vec{v}_j \wedge \hat{e}_3}{(\vec{v}_1 \wedge \vec{v}_2) \cdot \hat{e}_3}, \quad i = 1, 2, \quad (9)$$

where ϵ_{ij} is the totally anti-symmetric tensor. The reciprocal lattice vectors thus satisfy $\vec{g}_i \cdot \vec{v}_j = \delta_{ij}$. Now, let us turn to discuss the momentum transfers, which are obtained as the matrix elements of the interaction $\mathbf{V}(\mathbf{r})$ in the basis of plane-wave states with $\langle \mathbf{r} | \mathbf{k}, \alpha \rangle = e^{i\mathbf{k}\cdot\mathbf{r}} \otimes |\alpha\rangle$ for the spin component α . One finds that the matrix elements depend only on the momentum transfer $\Delta\mathbf{k} = \mathbf{k}' - \mathbf{k}$

$$V_{\mathbf{k}',\mathbf{k}}^{\alpha'\alpha} = V_{(\mathbf{k}'-\mathbf{k})}^{\alpha'\alpha} = \langle \mathbf{k}', \alpha' | \mathbf{V}(\mathbf{r}) | \mathbf{k}, \alpha \rangle. \quad (10)$$

According to Bloch's theorem, eigenstates $|\Psi_{n\mathbf{q}}\rangle$ are uniquely labelled by a band index n and momentum \mathbf{q} in the first BZ, while larger momenta can be decomposed as $\mathbf{k} = \mathbf{q} + \mathbf{G}$ into a part lying in the BZ and a reciprocal lattice vector $\mathbf{G}_{st} = s\vec{g}_1 + t\vec{g}_2$ with s, t integer. In its Bloch form the wavefunction reads

$$|\Psi_{n\mathbf{q}}\rangle = \sum_{\alpha} u_{n\mathbf{q}}^{\alpha}(\mathbf{r}) | \mathbf{q}, \alpha \rangle \equiv \sum_{\alpha, s, t} c_{n\mathbf{q}, \mathbf{G}_{st}}^{\alpha} | \mathbf{q} + \mathbf{G}_{st}, \alpha \rangle, \quad (11)$$

with expansion coefficients $c_{n\mathbf{q}, \mathbf{G}}^{\alpha}$. As was noted previously [69], the flux lattice Hamiltonian takes the form of a tight-binding model in reciprocal space in which the kinetic energy plays the role of a harmonic confinement:

$$\hat{\mathcal{H}}_{\mathbf{q}} = \sum_{\alpha, \mathbf{G}} \frac{\hbar^2(\mathbf{q} + \mathbf{G})^2}{2m} \hat{a}_{\alpha, \mathbf{q} + \mathbf{G}}^{\dagger} \hat{a}_{\alpha, \mathbf{q} + \mathbf{G}} + \sum_{\alpha\alpha', \mathbf{G}\mathbf{G}'} V_{\mathbf{G}'-\mathbf{G}}^{\alpha'\alpha} \hat{a}_{\alpha', \mathbf{q} + \mathbf{G}'}^{\dagger} \hat{a}_{\alpha, \mathbf{q} + \mathbf{G}}, \quad (12)$$

written here in terms of the annihilation (creation) operators $\hat{a}_{\alpha, \mathbf{k}}^{(\dagger)}$ for the plane-wave basis. We should also carefully note that all hoppings in this momentum-space tight-binding representation are relative to the wave-vector \mathbf{q} , hence they represent a lattice of achievable *momentum transfers*, while in the usual case of tight-binding models in real space one is used to consider a lattice of fixed positions.

The depth of the optical lattice potential is reflected by the magnitude \mathcal{V} of the largest entries in $V_{\Delta\mathbf{k}}^{\alpha'\alpha}$. The typical kinetic energy is of order of the recoil energy, which we define in terms of the relevant momentum transfer $\Delta\mathbf{p} = \hbar\Delta\mathbf{k}$ of the relevant laser-beam as

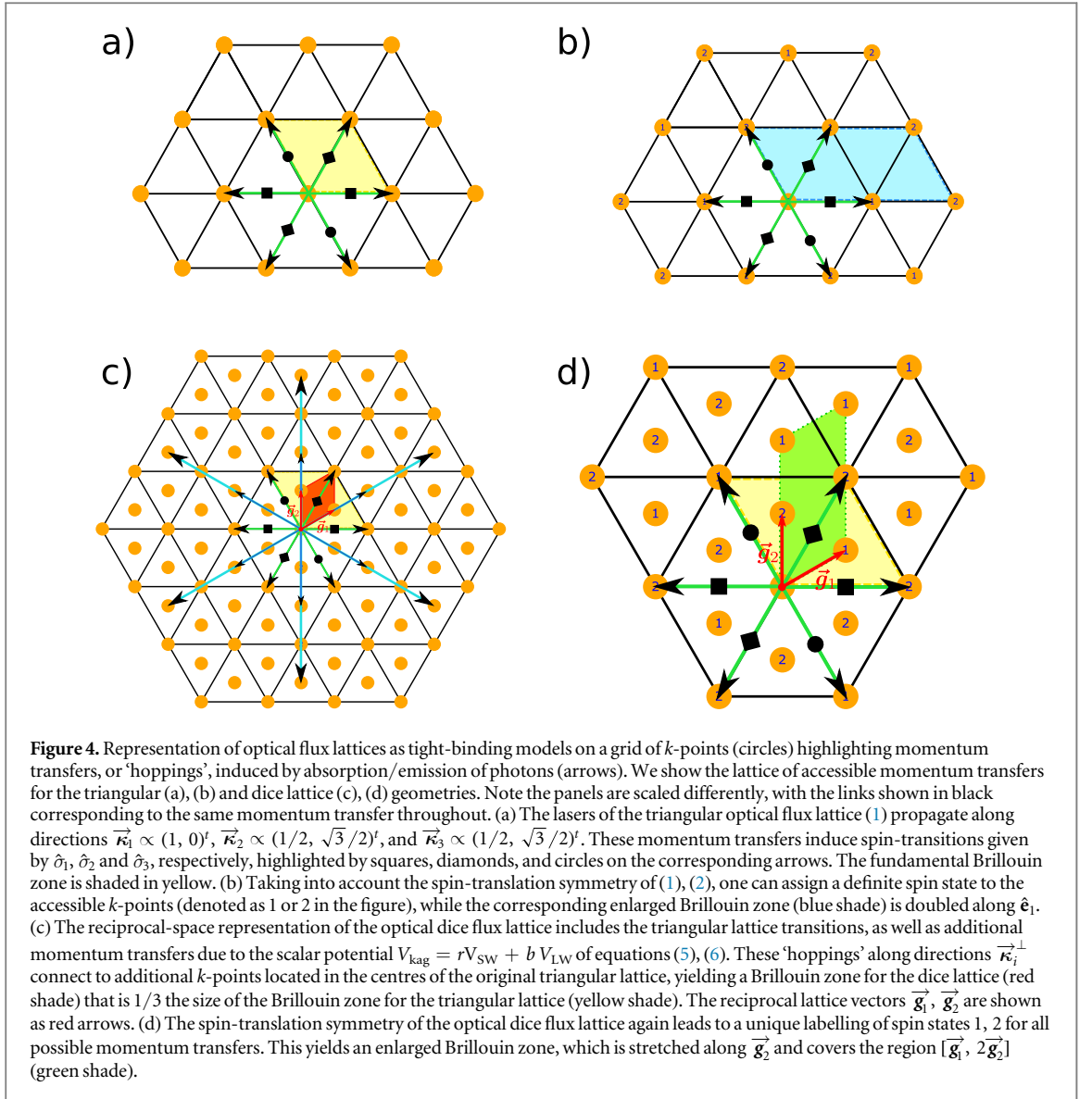
$$E_R = \frac{\hbar^2 |\Delta\mathbf{k}|^2}{2m}. \quad (13)$$

The adiabatic limit is recovered when $E_R \ll \mathcal{V}$, where the kinetic energy can be neglected and the problem is solved by Fourier transform back into real space, where position \mathbf{r} plays the role of a conserved momentum. In the general case, (12) defines a matrix equation for the coefficients $c_{n\mathbf{q}, \mathbf{G}}^{\alpha}$, which can be solved numerically as coefficients decay rapidly with the absolute value of momentum $|\mathbf{q} + \mathbf{G}|$.

5.2. Role of spin-translation symmetries in complex optical flux lattices

The role of the spin-translation symmetries is more easily explained within an example. Let us therefore focus on the reciprocal space picture of the dice flux lattice $\mathcal{H}_{\text{dice}}$, that is illustrated in figure 4. For the components associated with the triangular flux lattice (1), we obtain the spin-dependent processes $\hat{V}_{\vec{\kappa}_1} = \mathcal{V}\hat{\sigma}_1$ with momentum transfer $\vec{\kappa}_1 = -\vec{g}_1 + 2\vec{g}_2$, $\hat{V}_{\vec{\kappa}_2} = \mathcal{V}\hat{\sigma}_2$ with $\vec{\kappa}_2 = \vec{g}_1 + \vec{g}_2$, and $\hat{V}_{\vec{\kappa}_3} = \mathcal{V}\hat{\sigma}_3$ with $\vec{\kappa}_3 = 2\vec{g}_1 - \vec{g}_2$, where the reciprocal lattice vectors \vec{g}_i are defined by the lattice vectors \vec{v}_i spanning the unit cell of the dice flux lattice according to (4). For later reference, note that these momentum transfers are proportional to the wave vectors of the three coupling lasers of the dice optical flux lattice, and are linear combinations in integer multiples of its reciprocal lattice vectors (9).

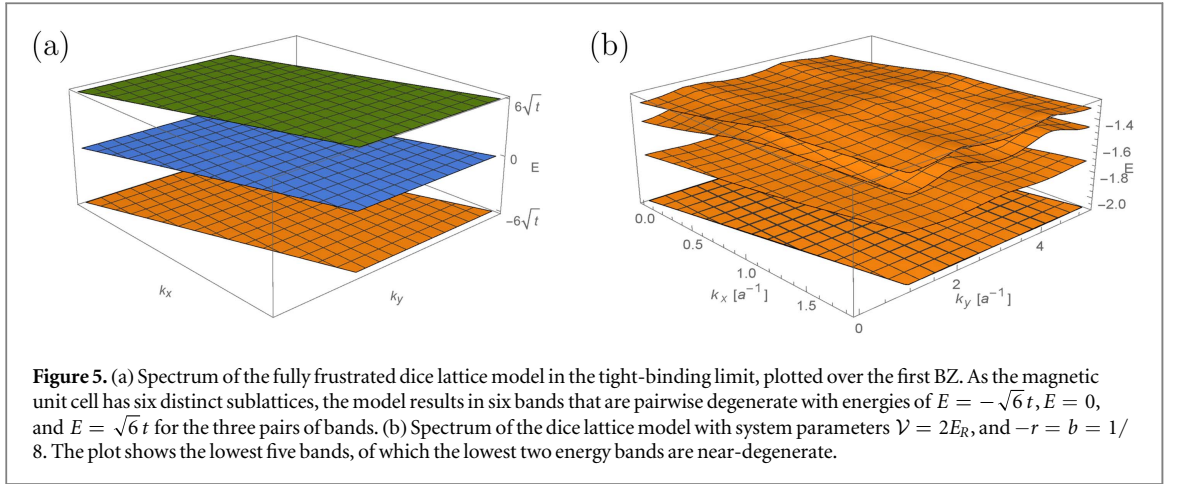
We display the momentum transfers of the underlying triangular optical flux lattice in figure 4(a), which also highlights the BZ corresponding to full the real-space unit cell $[\vec{a}_1, \vec{a}_2]$ of figure 3. Following [69], the spin-translation symmetry \hat{T}_1 of this model can be exposed by fixing the eigenvalue of the spin-translation operator,



leading to a halving of the real space unit cell to $[\vec{a}_1/2, \vec{a}_2]$, thus doubling the BZ and leaving a definite spin state at each reciprocal lattice site, as shown in figure 4(b).

To obtain the dice optical flux lattice, we add to this picture the coupling to the scalar optical potentials generating the kagome lattice, equations (5), (6), which contribute with momentum transfers corresponding to *twice* their wave numbers, arising from the absorption of a photon from a standing wave laser followed by stimulated emission in the opposite direction. For V_{SW} , we obtain momentum transfers $\Delta \vec{k}_i^{\text{SW}} = 2 \vec{\kappa}_i^\perp$, with amplitude $\hat{V}_{\text{SW}} = r\hat{V}$ and similarly for V_{LW} the momentum transfers are $\Delta \vec{k}_i^{\text{LW}} = \vec{\kappa}_i^\perp$ with amplitude $\hat{V}_{\text{LW}} = b\hat{V}$. These momentum transfers are four- or twofold multiples of the reciprocal lattice vectors and their $\pi/3$ rotations.

According to the enlarged unit cell in real space, the BZ of the dice lattice should cover one third of the area of the BZ for the triangular optical flux lattice. The corresponding lattice of possible momentum transfers is illustrated in figure 4(c), revealing a three times denser coverage of attainable k -points. The action of the spin-translation symmetry of the dice lattice model is again readily illustrated in this momentum-space picture. Assume a single-particle wavefunction has a non-zero amplitude for spin state 1 and vanishing amplitude for spin state 2 at momentum \mathbf{q} . Applying momentum- and spin-transfers to this initial state according to the tight-binding Hamiltonian (12), one can see that all related reciprocal lattice points at positions $\mathbf{q} + \mathbf{G}$ are reached with a definite spin quantum number. Equivalently, the Hamiltonian does not allow one to create any loops that return to the initial point with a different value of the spin. Choosing a spin state of 1 at the central k -point, one



obtains the spin labels shown in figure 4(d). An equivalent labelling is obtained by interchanging labels ‘1’ and ‘2’ (or equivalently, by a translation of the figure under \vec{g}_2).

The spin-translation symmetry can be more formally derived from the eigenvalue equations of the spin-translation operators $\hat{S}_{1,2}$. We take \hat{S}_2 and \hat{S}_1^2 as the chosen symmetry generators commuting with the Hamiltonian, or $[\hat{\mathcal{H}}, \hat{S}_1^2] = [\hat{\mathcal{H}}, \hat{S}_2] = [\hat{S}_1^2, \hat{S}_2] = 0$, as discussed in section 4. This implies that the Hamiltonian is block-diagonal in the subspaces of fixed eigenvalues of \hat{S}_1^2, \hat{S}_2 . Given the unitarity of these operators, we denote their eigenvalues as $\lambda_i = \exp(i\Theta_i)$, with $\hat{S}_1^2|\Theta_1, \Theta_2\rangle = \exp(i\Theta_1)|\Theta_1, \Theta_2\rangle$ and $\hat{S}_2|\Theta_1, \Theta_2\rangle = \exp(i\Theta_2)|\Theta_1, \Theta_2\rangle$, with $|\Theta_1, \Theta_2\rangle$ the corresponding eigenstates. Consider then the explicit action of the generalised translations on momentum eigenstates

$$\begin{aligned}\hat{S}_1^2|\mathbf{k}, \alpha\rangle &= \hat{\mathbf{1}}e^{i\vec{v}_1\cdot\mathbf{k}}|\mathbf{k}, \alpha\rangle, \\ \hat{S}_2|\mathbf{k}, \alpha\rangle &= i\hat{\sigma}_3 e^{i\frac{1}{2}\vec{v}_2\cdot\mathbf{k}}|\mathbf{k}, \alpha\rangle.\end{aligned}\quad (14)$$

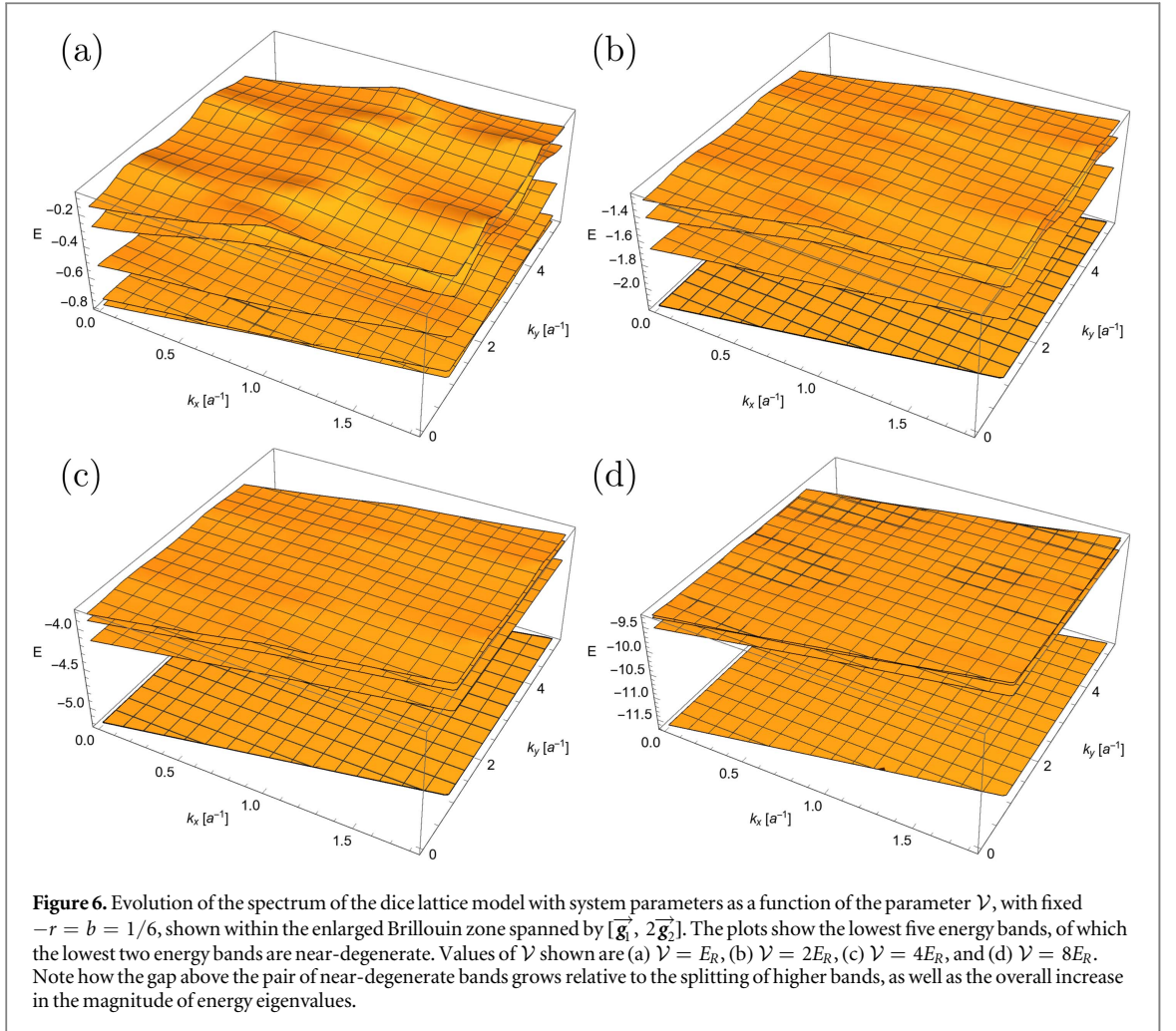
We see that the phases are periodic under translations of $\mathbf{k} \rightarrow \mathbf{k} + \vec{g}_1$ in the phase of \hat{S}_1^2 , while the action of \hat{S}_2 is periodic under a doubled reciprocal lattice vector $\mathbf{k} \rightarrow \mathbf{k} + 2\vec{g}_2$, when the spin state is fixed. Thus, we can label eigenstates by a momentum \mathbf{q} taken to lie in the enlarged BZ $[\vec{g}_1, 2\vec{g}_2]$ that is stretched twofold along the \vec{g}_2 -direction, as highlighted in figure 4(d). In this representation, each point of reciprocal momentum transfers can be assigned a definite spin state, as the momentum \mathbf{q} in the enlarged BZ provides sufficient information to encode both the spin and momentum degrees of freedom. Alternatively, one could choose to represent the full range of possible eigenvalues $\Theta_2 \in [0, 2\pi)$ by reducing the momentum to the fundamental BZ $[\vec{g}_1, \vec{g}_2]$, and recover the full range of Θ_2 by taking into account both ± 1 eigenvalues of the spin operator $\hat{\sigma}_3$.

6. Quantitative analysis of the π -flux optical dice flux lattice

In this section, we provide a numerical study of the optical dice flux lattice introduced in section 5. Our numerics are performed in terms of the reduced unit cell $[\vec{v}_1, \vec{v}_2/2]$, or its reciprocal space counterpart. In other words, our implementation relies on resolving eigenstates of the generalized translations \hat{S}_2 , as discussed above.

In practice, we diagonalize the matrix representing the Hamiltonian (12) in the basis of the plane-wave states selected by the spin-translation symmetry, as shown in figure 4(d). We include basis states at wave vectors $\mathbf{q} + \mathbf{G}_{s,b}$ with $|s|, |t| \leq t_{\max}$. We find that values of t_{\max} ranging up to 16 are sufficient to achieve good convergence for the cases shown below. Diagonalisation of the ensuing matrix yields the desired spectra and eigenstates $|\Psi_{n\mathbf{q}}\rangle$. We obtain these on a discrete grid of momenta \mathbf{q} within the extended BZ spanned by $[\vec{g}_1, 2\vec{g}_2]$.

We proceed to discuss the spectrum, which provides an excellent approximation to the tight-binding version of the π -flux dice lattice model. For reference, let us first review the spectrum in the tight-binding limit, shown in figure 5(a). Note that the tight-binding spectrum features only three distinct eigenvalues, each corresponding to a pair of degenerate bands all of which are time-reversal symmetric and have Chern number $C = 0$. The overall count of six bands corresponds to the six lattice sites in the fundamental magnetic unit cell of the fully frustrated dice lattice.

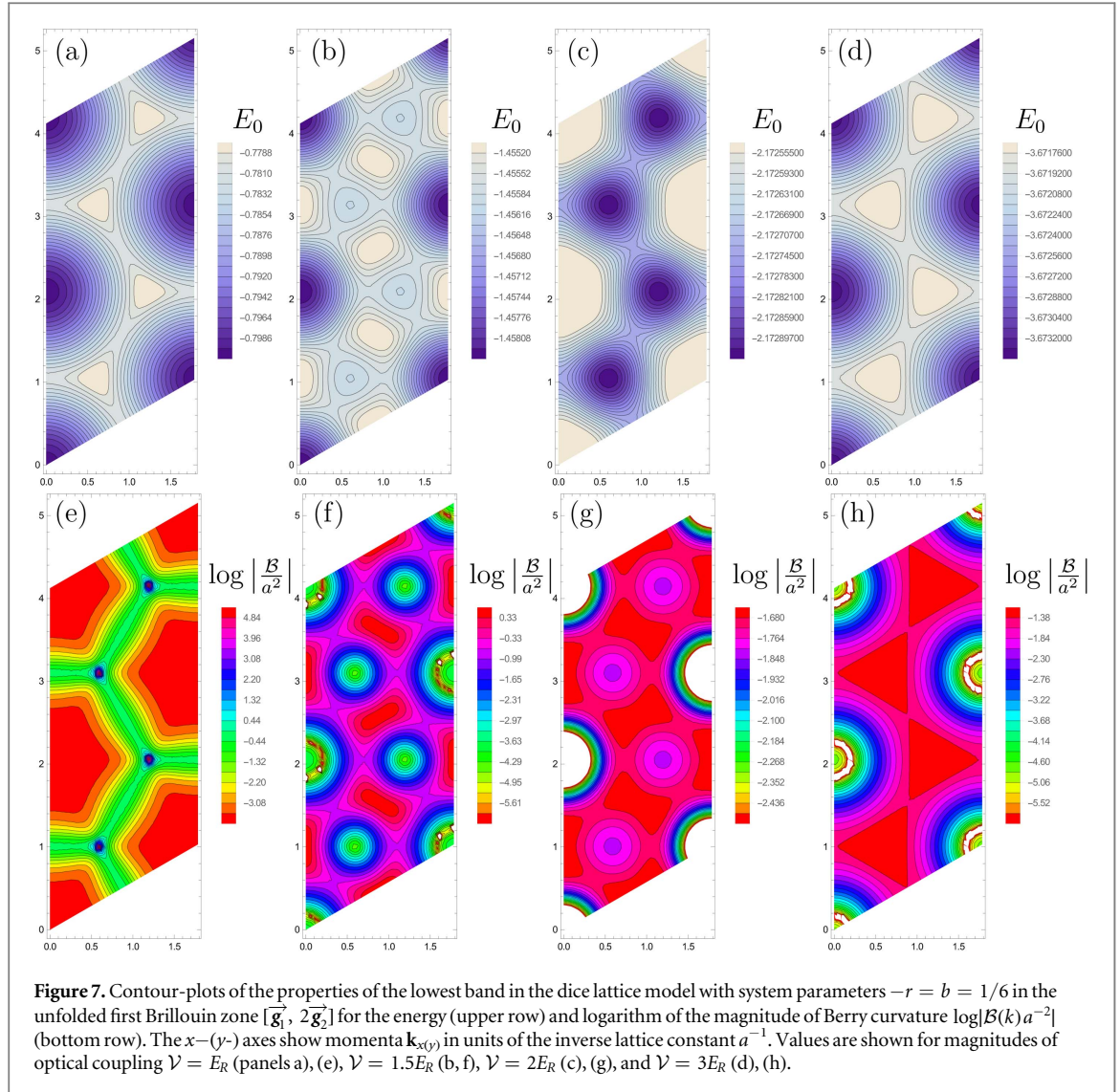


At intermediate-depth of the optical lattice $\mathcal{V} \simeq E_R^4$, we find that the low-energy spectrum of our proposed dice flux lattice (4) correctly reproduces the qualitative features of the tight-binding model. For $\mathcal{V} \simeq E_R$, this low-energy spectrum contains two near-degenerate bands that are well separated from higher bands. These two lowest bands have a very small dispersion and have only a small residual splitting. A typical spectrum, for $\mathcal{V} = 2E_R$, and $-r = b = 1/8$ is shown in figure 5(b). To display the residual dispersion of the lowest bands more clearly, we will analyse a series of contour-plots in figure 7, below. For the parameters in figure 5(b), the dispersion of the two lowest bands is of the order of $0.04E_R$. There is a small splitting to the second band (not shown), which has the inverse dispersion relative to that of the lowest band, i.e. its minima are found at the maxima of the lowest band and vice versa. With these parameters, the joint dispersion of these nearly degenerate bands is about 50 times smaller than the gap to higher excited bands.

It is instructive to analyse how the band dispersion evolves with the strength \mathcal{V} of the optical coupling. A series of different spectra with values ranging from $\mathcal{V} = E_R$ to $\mathcal{V} = 8E_R$ is shown in figure 6, including the lowest five bands in each case. These data were obtained with a cut-off for momentum at $k \simeq 12|\vec{g}_1|$. It is clearly seen that the near-degeneracy of the lowest two bands is realised very well for all $\mathcal{V} \geq 2E_R$, while a small splitting is visible on the figure for $\mathcal{V} = E_R$. The higher ($n = 3, 4, 5$) bands are not found to be degenerate. However, the gap above the near-degenerate ground-state manifold is seen to increase with the optical coupling strength. Given these findings, we interpret the lowest bands as corresponding to the two degenerate lowest energy bands in the tight-binding limit, while the higher bands can be interpreted as arising from different local orbitals that can be formed within the minima of the optical potential. In the limit of $\mathcal{V} \rightarrow \infty$, we expect that the splitting to such orbitals would become large, and a low-energy part corresponding to the single orbital physics may then emerge from the spectrum.

We now discuss the topological nature of the low-lying bands in the dice flux lattice. The main qualitative difference of the intermediate-depth lattice with respect to the tight-binding model is the occurrence of weak

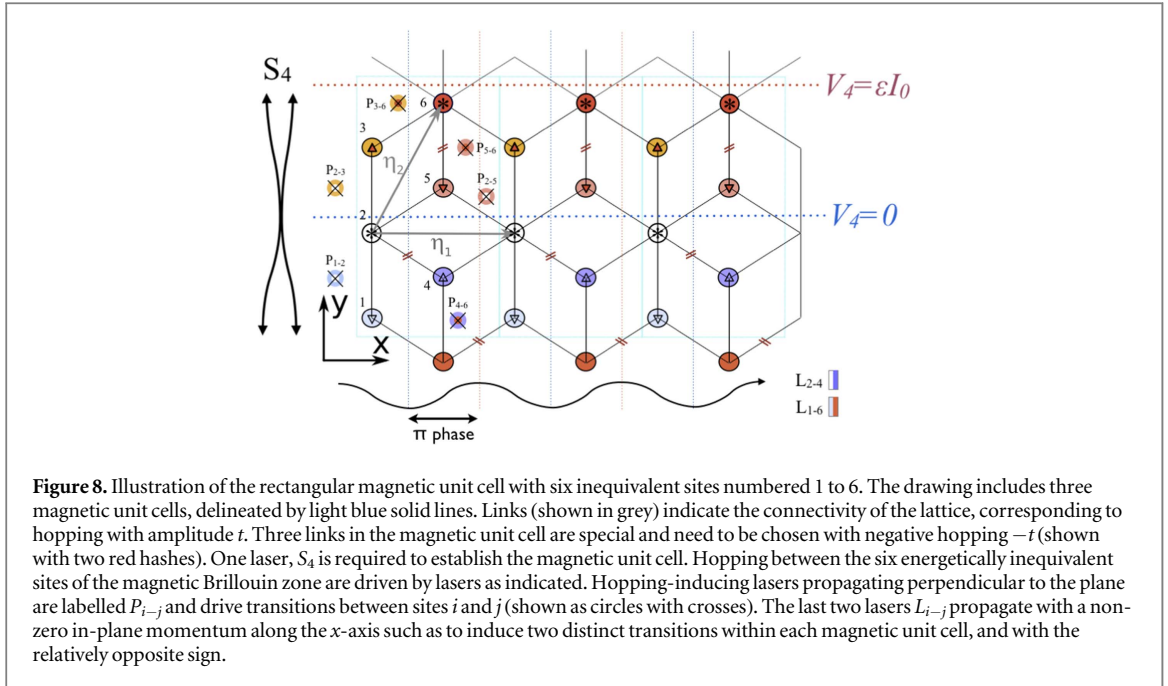
⁴ Here, we define the recoil energy E_R as in (13), using the wave number κ for the underlying triangular lattice as the reference. Although this is not the largest momentum transfer in the set-up, it is the laser requiring the largest amplitude.



tunnelling across the ‘forbidden’ links of the underlying triangular flux lattice, which break time-reversal symmetry. To analyse this statement quantitatively, we calculate the Berry curvature $\mathcal{B}(\mathbf{q}) = i\epsilon_{\mu\nu} \frac{\partial}{\partial q_\mu} \langle \Psi_{0,\mathbf{q}} | \frac{\partial}{\partial q_\nu} | \Psi_{0,\mathbf{q}} \rangle$ for our model, where $\epsilon_{\mu\nu}$ is the totally anti-symmetric tensor and Einstein’s summation convention is implied. To implement \mathcal{B} computationally without the need to explicitly evaluate the derivatives in this expression, we have instead evaluated the integral of the Berry curvature over small Wilson loops on a discretized grid of k -points within the BZ following Fukui *et al* [70]. We confirm that the Berry curvature is non-zero, and has opposite signs in the two low-lying bands. The distributions of the (log-) Berry curvature in the lowest band are shown as contour-plots in the lower row of figure 7 for a range of optical coupling strengths, while the upper row shows the corresponding band dispersions. Note that there are extended regions where the curvature \mathcal{B} is small, while maxima are relatively localised. For example, at $\mathcal{V} = E_R$, typical values are $\mathcal{B} \simeq 0.05a^2$ (to be compared to an average of $\bar{\mathcal{B}} = 9/\pi Ca^2 \simeq 2.86Ca^2$ for a Chern number \mathcal{C} band with homogenous Berry curvature of the given BZ area). At the location of the maxima of the band dispersion, which can be seen as avoided crossings with the next higher band, \mathcal{B} is strongly peaked and as a result, the Chern number \mathcal{C} of the band is non-zero. Depending on the specific parameters we have found either $|\mathcal{C}| = 1$, or $|\mathcal{C}| = 3$. In both cases, the cumulative Chern number of the two lowest bands is zero.

The different panels of figure 7 show the evolution of the band dispersion with increasing optical coupling, which reveals a change of the location of minima in the dispersion, and correspondingly for the Berry curvature. Note also how the flatness of the bands improves as we go to stronger coupling. Extended regions of low Berry curvature are also found at the highest value we show.

It would be interesting to study how the many-body spectrum is affected by this finite but oppositely oriented Berry curvature in the lowest two bands. We expect that as long as the interaction energy is larger than the residual splitting between the two lowest bands, the system likely behaves in a qualitatively similar fashion as



the time-reversal invariant system in the tight-binding limit [63]. A detailed analysis of this physics will be the subject of a future study. In the sense that the perturbation of the bands away from the time-reversal symmetric case is caused by small hopping elements on suppressed bonds, we can consider the time-reversal symmetry breaking of our optical dice flux lattice to be ‘weak’.

7. Realizing the fully frustrated dice lattice in a tight-binding approach

An alternative realisation of the dice lattice pierced by π -flux per plaquette can be realised in a pure tight-binding philosophy. Let us discuss in detail the set-up for alkaline earth atoms (e.g., ytterbium (Yb)) atoms trapped in an optical lattice at the anti-magic wavelength [63]. In our approach, we closely follow the proposal for a square optical lattice using anti-magic trapping [22]. The possibility for this construction arises as the two internal states (1S_0 and 3P_0) of Yb have polarisability α of opposite signs for wavelengths $\lambda \gtrsim 960$ nm, so they are trapped at the points of maximum or minimum laser intensity, respectively [22]. At the anti-magic wavelength $\lambda^* \simeq 1120$ nm, the absolute values of the polarisability are of equal magnitude. This is crucial for the square lattice geometry. For our purposes, it may actually be more useful to choose a wavelength at which the polarisability is stronger in magnitude for one of the two (pseudo-)spin states: the dice lattice geometry results from a triangular optical lattice formed by three self-reflected laser beams propagating with wave vectors arranged at relative angles of $2\pi/3$ with respect to each other. These beams should be mutually incoherent, so the total intensity is the sum of individual intensities. The mirrors used to self-reflect these beams need to be stabilised⁵. One species of atoms (1S_0) is then trapped at the maxima of the intensity (which are steep), while the excited 3P_0 state is trapped at the minima (which are more shallow). Hence, it is favourable that the polarisability is larger in magnitude for the excited state, implying use of a wavelength $\lambda_0 \equiv 2\pi/k_0 > \lambda^*$, i.e. using wavelengths in the far infrared (given that the polarisability of the excited state grows more rapidly with λ near the anti-magic wavelength, or $d\alpha(^3P_0)/d\lambda|_{\lambda^*} > d\alpha(^1S_0)/d\lambda|_{\lambda^*}$).

In our set-up, all neighbouring sites are occupied by atoms of different internal states. Consequently, spontaneous tunnelling processes can be neglected, and all dynamics in this lattice is driven by via laser-assisted hopping [17, 71]. Simultaneously, this coupling enables one to imprint phases onto the hopping matrix elements [22]. Let us now explain how to achieve phases that yield the target flux density of $n_\phi = 1/2$. For the fully frustrated dice lattice, the magnetic unit cell contains six inequivalent atoms [61, 63], chosen here as a rectangular cell spanned by vectors $\vec{v}_1 = (\sqrt{3}a, 0)^t$ and $\vec{v}_2 = (0, 3a)^t$, as indicated by the different colouring of inequivalent lattice sites in figure 8. However, the (scalar) triangular optical lattice described in the preceding paragraph distinguishes only two types of sites. We propose to break this symmetry by shining one additional self-reflected laser-beam, S_4 , onto the system: this beam serves to break down the internal mirror-symmetry of

⁵ Alternatively, a suitable triangular lattice potential can be generated by three running beams with relative phase coherence. However, these would additionally have to be phase stabilised to prevent this triangular lattice from drifting relative to the 4th standing wave, laser S_4 , discussed below.

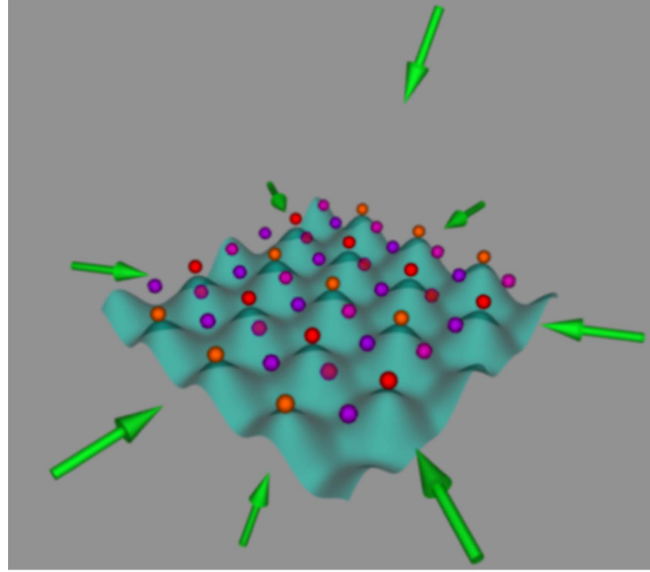


Figure 9. Set up of an optical dice lattice using an anti-magic optical lattice with laser-induced hopping: an underlying triangular lattice is created by retro-reflected standing wave lasers in plane. The symmetry of the magnetic unit cell is created by an additional standing wave laser S_4 directed at an angle to the plane. Eight coupling lasers complete the set-up and drive transitions between sites of different energy, as shown in figure 8 and discussed in the main text.

the triangular lattice unit cell to the desired periodicity. In our set-up, S_4 has the same frequency/wavelength as the triangular optical lattice. However, its in-plane wavelength is enlarged to $\lambda_4^{\parallel} = \lambda_0 / \sin(\theta)$ by projecting this laser onto the system at a tilt angle θ with respect to the z -axis of the plane. We tilt the laser towards the y -direction and require the potential to repeat on the scale of the magnetic unit cell, i.e., $|\vec{v}_2^{\parallel}| = \lambda_4^{\parallel}/2$. By geometry, we must therefore choose the angle $\theta = \arcsin(1/2) = \pi/6$. Note the position of this laser potential (S_4) needs to maintain a fixed spatial position relative to the lasers defining the optical lattice, as fluctuations would shift the superlattice potential relative to the triangular lattice potential, and would alter the relative magnitudes of site energies. However, these energies need to be precisely defined, so that coupling lasers can satisfy the resonance condition and match the binding energy differences for the links on which they induce hopping processes. Note that a different wavelength laser could also be chosen.

To be explicit, let us write the required laser potentials. A bare triangular optical lattice of lattice constant a is created by the wavelength $\lambda_0 = 3a$ of the trapping beams:

$$V_{\text{tri}}(\mathbf{r}) = I_0 \sum_i \sin^2 \left(\frac{2\pi}{3a} \vec{\kappa}_i \cdot \mathbf{r} \right), \quad (15)$$

with the unit lattice directions $\vec{\kappa}_1 = (0, 1, 0)^t$, $\vec{\kappa}_2 = (-\sqrt{3}/2, 1/2, 0)^t$, and $\vec{\kappa}_3 = (\sqrt{3}/2, 1/2, 0)^t$. The additional self-reflected laser, S_4 , propagates along the direction $\vec{n}_d = (0, \sin \theta, \cos \theta)^t$, adding an (in-plane) intensity distribution of

$$V_4(\mathbf{r}) = \epsilon I_0 \sin^2 \left(\frac{2\pi}{3a} \vec{n}_d \cdot \mathbf{r} + \delta \right) \equiv \epsilon I_0 \sin^2 \left(\frac{2\pi}{6a} y + \delta \right) \quad (16)$$

Here, we need to choose a small offset of the phase δ such that the maximum of intensity of the additional laser does not align with any high-symmetry point in the magnetic unit cell, and the intensity of the inversion symmetry breaking laser S_4 is reduced with respect to the other lasers by a suitable small factor ϵ , e.g., we can choose number of the order $\delta \simeq 2\pi/10$ and $\epsilon \simeq 0.05$.

A three-dimensional view of the overall set-up is given in figure 9. In the resulting potential $V_{\text{tot}}(\mathbf{r}) = V_{\text{tri}}(\mathbf{r}) + V_4(\mathbf{r})$, the six sublattices of the desired magnetic unit cell are all distinguished energetically, i.e. their energies being detuned with respect to the triangular lattice by distinct amounts $\delta\epsilon_i$, $i = 1, \dots, 6$.

The set-up is completed by a total of eight coupling lasers driving the respective transitions between these sites. All of these lasers are propagating waves. Six of them are directed onto the system in the direction perpendicular to the lattice-plane. We denote these lasers as P_{i-j} , indicating the two lattice sites i, j between which they induce a resonant transition. The six required lasers are P_{1-2} , P_{2-3} , P_{2-5} , P_{3-6} , P_{4-6} , and P_{5-6} , which require frequencies $\hbar\omega_{i-j} = \hbar\omega_i^0 + \delta\epsilon_i - (\hbar\omega_j^0 + \delta\epsilon_j)$, and $\hbar\omega_i^0$ denotes the unperturbed energy of the internal state trapped at site i . Note each laser drives a transition between two neighbouring sites where atoms are in their ground / excited state, respectively. See also figure 8 for an illustration. Four of these six lasers drive a transition

on a single link in the unit cell. However, the two lasers P_{2-5}, P_{3-6} connect the sixfold connected sites ‘2’ and ‘6’ to two neighbours with identical energy, located in the same and an adjacent unit cell, respectively. Due to the perpendicular direction of the lasers, these transitions are driven in phase, so the hopping elements have the same sign. All but one of the lasers P_{i-j} need to be in phase with each other, while P_{5-6} requires a phase-shift of π relative to the others. A definite phase relationship between these lasers of different frequencies can be achieved by deriving them from a single light source, and detuning their frequency using an acousto-optic modulator. The remaining coupling two lasers, which we call L_{1-6} and L_{2-4} are special in that they are required to drive two transitions (like P_{5-6}), but now with a relative phase of π between these two couplings. This relative phase is realised by virtue of an in-plane component of the respective wave vectors. Specifically, we choose the in-plane component of their respective wave vectors \mathbf{k} along the x -axis such that $\mathbf{k} \cdot (\sqrt{3}a/2, 0, 0)^t \equiv \pi$. Again, this wave-vector can be realized by a suitable inclination of the laser beams with respect to the plane.

This concludes our discussion of the detailed set-up for a tight-binding version of fully frustrated dice lattice. Let us briefly compare this construction to the optical dice flux lattice discussed in section 6. Firstly, we note that the tight-binding construction is explicitly time-reversal invariant, if all relative phases are set to match the values 0 or π . Although there may be small perturbations to the ideal dice lattice model from spontaneous tunnelling processes between neighbouring threefold sites such as sites 1 and 4, such processes also have real hopping elements.

The practical realisation of both schemes poses similar challenges, notably the requirement to generate superlattice potentials whose relative position must be stabilised relative to an underlying lattice. This is difficult, but has already been achieved [54]. However, fluctuations of the geometry will affect the two proposals rather differently. In the optical flux lattice set-up, the superlattice acts to suppress tunnelling by creating local maxima in the potential. This suppression will be relatively insensitive to the precise location of potential maxima, as long as they are located within the relevant bonds of the lattice. By contrast, the tight-binding approach requires the superlattice to define relative energies of lattice orbitals, and transitions between them are driven resonantly. Hence, a rather fine control of the stability is required to ensure that all coupling lasers remain on resonance for their respective bonds.

8. Conclusions

We have introduced a new method for constructing optical flux lattices with complex geometries by combining a simple optical flux lattice with additional scalar potentials. To demonstrate the potential of our proposal, we have explored the optical dice flux lattice as an example geometry in which bonds were eliminated from an underlying triangular lattice. Our model yields flat bands that are a particularly interesting playground for studying interaction-driven phases of matter [63], and can realise a flatness parameter of fifty even for weak optical coupling. The optical flux lattice approach results in interesting additional features with respect to a pure tight-binding description of the dice lattice model. At intermediate lattice depths, the model weakly breaks time-reversal symmetry in the following sense: instead of degenerate pairs of time-reversal symmetric bands, the approach produces time-reversal pairs of bands whose degeneracies are only weakly split.

The proposed realisation of an optical dice flux lattice is realistically achievable in the near future, as it combines several elements which are already part of the current state of the art. The kagome lattice realised in the group of Stamper-Kurn successfully demonstrates the phase-stabilised superposition of two lattices with distinct wavelengths [54]. Our set-up requires the additional superposition of a triangular optical flux lattice. While work on the first realisation of such systems under way, we would like to underline that related schemes for synthetic gauge fields have already been successful [38, 39], and related schemes for emulating spin-orbit coupling in 2D systems have also been implemented [50–52].

We have also introduced a proposal for a tight-binding scheme which is closer to the existing technology of the aforementioned experiments. Here, challenges rely on fine-tuning energies and maintaining the relative superlattice position with high accuracy. This kind of set-up requires one-by-one engineering of laser-induced hopping between sites in the unit cell, so its complexity grows with the unit cell size.

By contrast, one of the inherent features of the flux lattice schemes is their tuneability. Explorations of scalar optical lattices have already shown that a multitude of different band-structures can be realised in the same experiment [54, 58]. Hence, one interesting direction for further study is the question of how the lattice geometry is altered when moving the scalar lattices with respect to the underlying optical flux lattice.

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