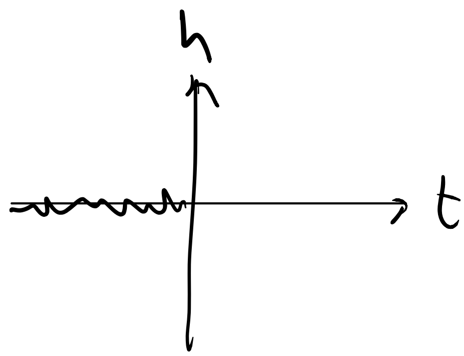
The scaling hypothesis



Recall from MFT

$$f = \min_{m} \left[\frac{t}{2} m^2 + u m^4 - h m \right]$$

$$\sim \begin{cases} -\dfrac{t^2}{u} & h = 0,\ t < 0 \\[2ex] -\dfrac{h^{4/3}}{u^{1/3}} & t = 0 \end{cases}$$

$$f(\alpha \underline{v}) = \alpha^k f(\underline{v}) \quad \text{homogeneous function}$$

Observation: f has homogeneous form.

$$f(t, h) = t^2 g_f\left(\frac{h}{t^\Delta}\right)$$

cf. mean-field

$$\lim_{x \to 0} g(x) \sim -\frac{1}{u} \quad \text{const.}$$

$$f(t, h=0) \sim -\frac{t^2}{u} \quad \checkmark$$

$$\lim_{x \to \infty} g(x) \sim x^{4/3}$$

$$f(t=0,h) \sim t^2 \cdot \left(\frac{h}{t^\Delta}\right)^{4/3} \to h^{4/3}$$

$\Delta = 3/2$ – gap exponent.

Assumption

The singular part of f has a homogeneous form

$$f_{sing}(t,h) = t^{2-\alpha}\, g_f\left(\frac{h}{t^\Delta}\right)$$

α, Δ unspecified

Magnetisation

$$m(t,h) = -\frac{\partial f}{\partial h} = -t^{2-\alpha-\Delta}\, g_f'\left(\frac{h}{t^\Delta}\right)$$

$$= t^{2-\alpha-\Delta}\, g_m\left(\frac{h}{t^\Delta}\right)$$

$$m(t,h=0) = t^{2-\alpha-\Delta}\, g_m(0) \sim t^\beta$$

if $\beta = 2-\alpha-\Delta$

$$m(t=0,h) = t^{2-\alpha-\Delta}\left(\frac{h}{t^{\Delta}}\right)^{p} \qquad g_m(x) \sim x^{p}$$

$$= t^{2-\alpha-\Delta-p\Delta}\, h^{p} \sim h^{\frac{2-\alpha-\Delta}{\Delta}} \sim h^{\frac{1}{\delta}}$$

if $p\Delta = 2-\alpha-\Delta$

and $\delta = \dfrac{\Delta}{2-\alpha-\Delta} = \dfrac{\Delta}{\beta}$

i.e. singular form of f fixes the singular form of m.

Susceptibility $\chi(t) = \dfrac{\partial m}{\partial h}$

$$= t^{2-\alpha-2\Delta}\, g_{\chi}\left(\frac{h}{t^{\Delta}}\right)$$

$$\sim t^{-(2\Delta+\alpha-2)} \sim t^{-\gamma}$$

if $\gamma = 2\Delta+\alpha-2$

Consequences of the homogeneity assumption

1. For all thermodynamic quantities $Q(t,h)$ exponents above and below T_c are the same.

2. All Q have the same gap exponent and α.

3. Almost all exponents can be obtained from 2 independent ones e.g. α and Δ.

Exponent Identities

$$\alpha + 2\beta + \gamma = \alpha + 2(2-\alpha-\Delta) + (2\Delta + \alpha - 2)$$
$$= 2$$

Rushbrooke

$$\delta - 1 = \frac{\gamma}{\beta}$$ Widom

$$\delta - 1 = \frac{\Delta}{2-\alpha-\Delta} - 1 = \frac{\Delta - (2-\alpha-\Delta)}{2-\alpha-\Delta}$$

$$= \frac{2\Delta + \alpha - 2}{\beta} = \frac{\gamma}{\beta}$$

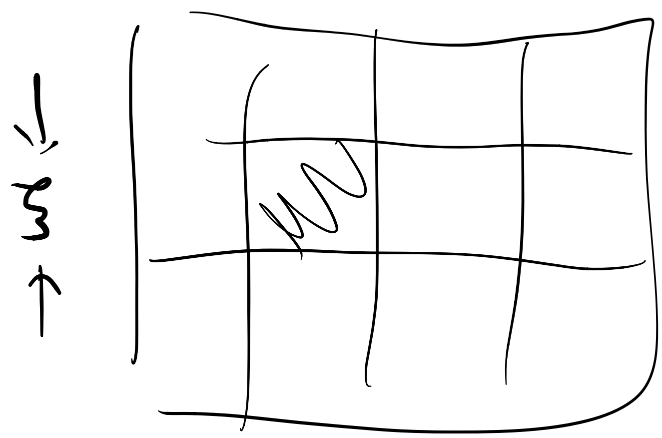
Hyperscaling and correlation length

1. Assume homogeneity of ξ

$$\xi(t,h) = t^{\nu} g_{\xi}\left(\frac{h}{t^{\Delta}}\right)$$

$$\xi(0,h) \sim h^{-\nu_h}, \text{ if } \nu_h = \frac{\nu}{\Delta}.$$

2. Close to T_c, ξ is only important length scale
   - i.e. determines singular part of thermodynamic quantities.



$$f_{sing} \sim \left(\frac{L}{\xi}\right)^d \times \text{"free energy of box"}$$

$$\therefore f_{sing}(t,h) \sim \xi^{-d} g_f\left(\frac{h}{t^{\Delta}}\right)$$

$$\sim t^{d\nu}\, \tilde{g}_f\left(\frac{h}{t^{\Delta}}\right)$$

1. Homogeneity of f follows from that of ξ.
2. Hyperscaling

$$\boxed{2-\alpha = d\nu} \qquad \text{Josephson identity}$$

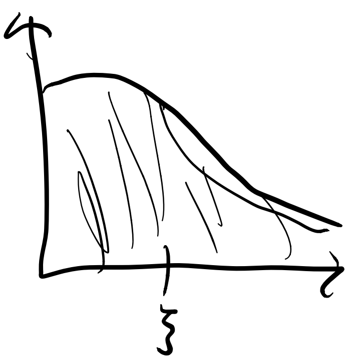
Critical correlation function + Self similarity

$$G(x,t) = \frac{1}{|x|^{d-2+\eta}}\, g_G\left(\frac{x}{\xi(t)}\right)$$

At critical point $(t = h = 0)$, $\xi \to \infty$

$$G(\underline{x}) = \langle \underline{m}(\underline{x})\, \underline{m}(\underline{0}) \rangle_c$$

$$\sim \frac{1}{|x|^{d-2+\eta}} \quad ; \quad S(q) \sim \frac{1}{|q|^{2-\eta}}$$



$$\chi \sim \int d^d\underline{x}\ \langle \underline{m}(\underline{x})\, \underline{m}(\underline{0}) \rangle_c$$

$$\sim \int_0^{\xi} d^d x \; \frac{1}{|x|^{d-2+\eta}}$$

$$\sim \xi^{2-\eta} \sim t^{-\nu(2-\eta)}$$

$$\gamma = \nu(2-\eta) \quad - \text{ Fisher}$$

<u>Consequences of scaling</u>

Critical systems have dilation symmetry

$$G_{\text{Critical}}(\lambda x) = \lambda^{p} G_{\text{crit}}(x)$$

$\Rightarrow$ scale invariance or self-similarity

— <u>fractal</u>

$\rightsquigarrow$ Conformal field theory (2d)

or <u>RENORMALISATION GROUP</u>

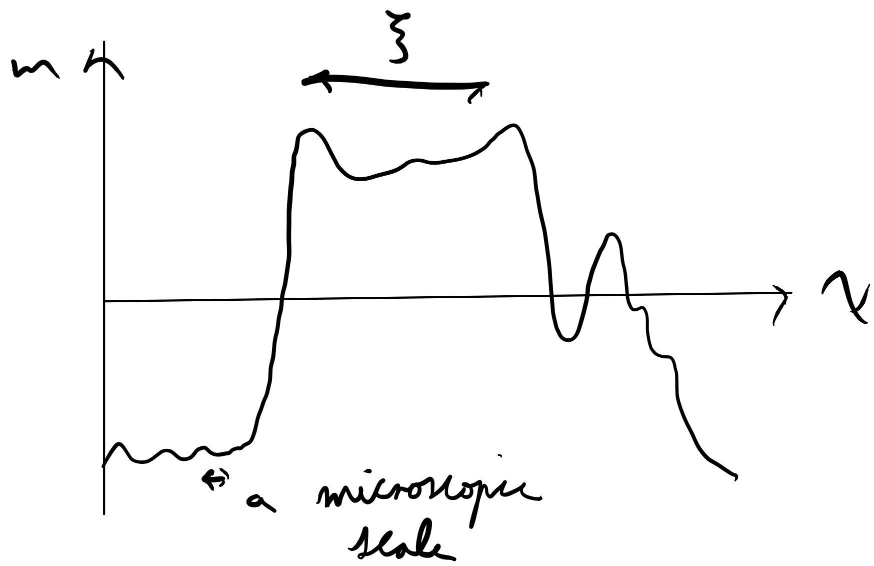
Kadanoff's Renormalisation Group (conceptual)

Suppose ξ is the only important length scale.

Renormalisation

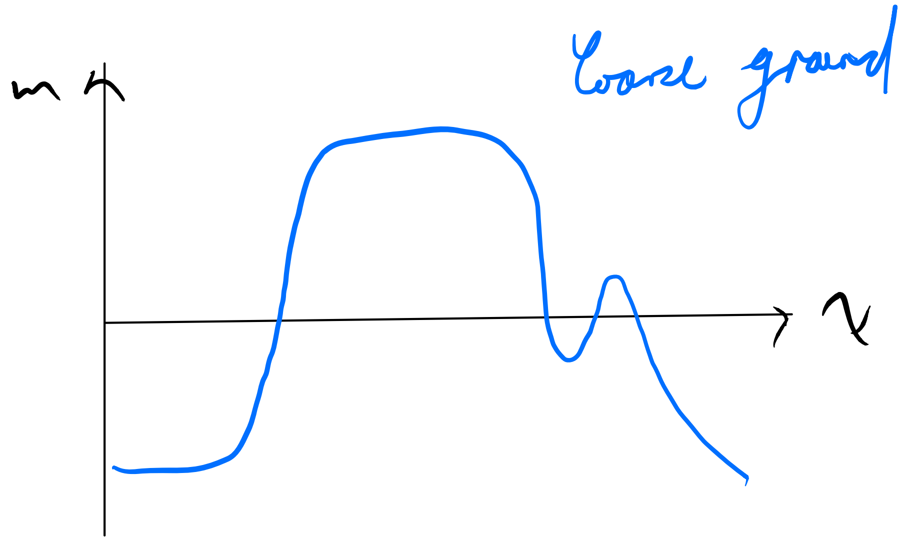
- eliminate short length scale fluctuations.

Start with a configuration $\underline{m}(\underline{x})$ with a Boltzmann weight $W[\underline{m}] = \exp[-\beta H[\underline{m}]]$



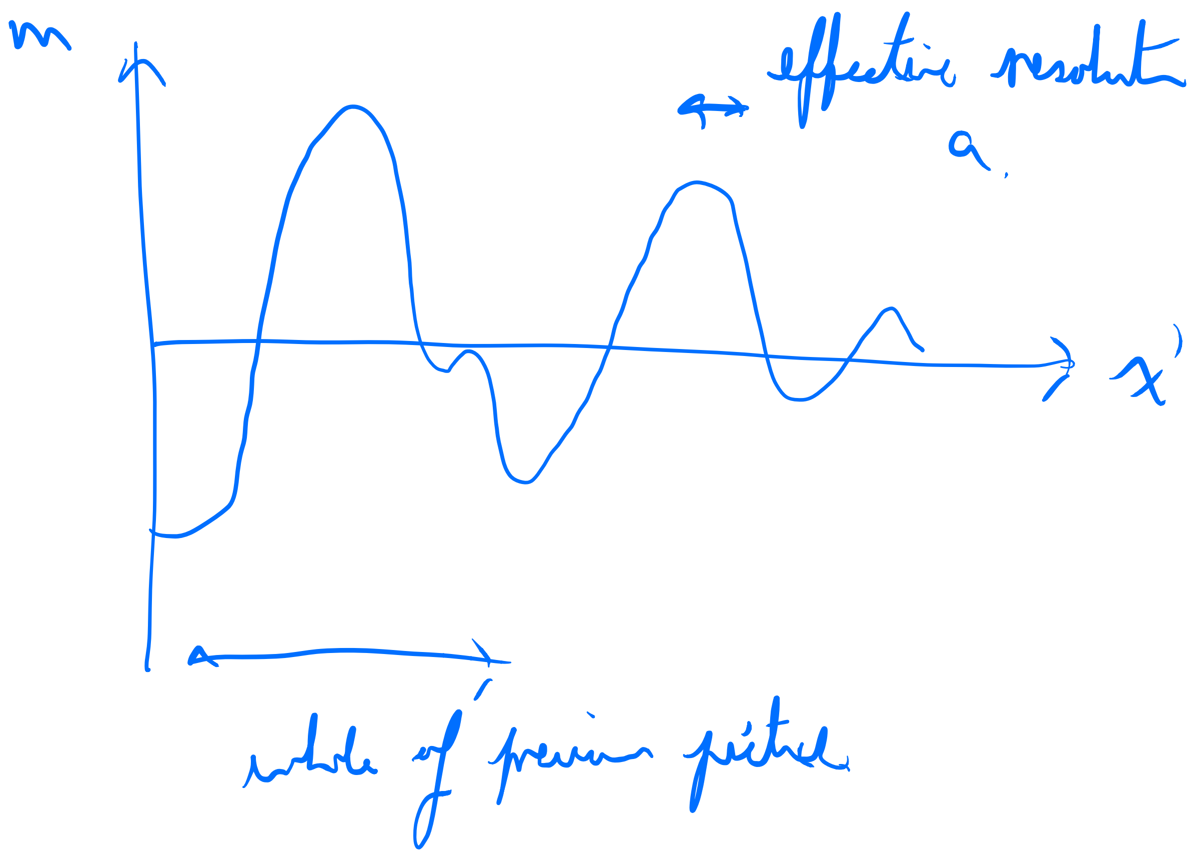
1) Coarse-grain by reducing resolution to ba $(b>1)$ with

$$\underline{m}(\underline{x}) = \frac{1}{(ba)^d} \int_{\substack{\text{cell centred at } \underline{x} \\ \text{of size } ba}} d^d y \; m(y)$$



2. Rescaling: "pixel is grown" – restore resolution by scaling.

$$x' = \frac{1}{b} x$$



3. Renormalize: Reducing contrast by factor ζ

$$\underline{m}'(\underline{x}') \equiv \frac{1}{\zeta}\,\underline{m}(\underline{x}')$$

$$\underline{m}(\underline{x}) \xrightarrow[\text{R.G.}]{} m'(\underline{x}')$$

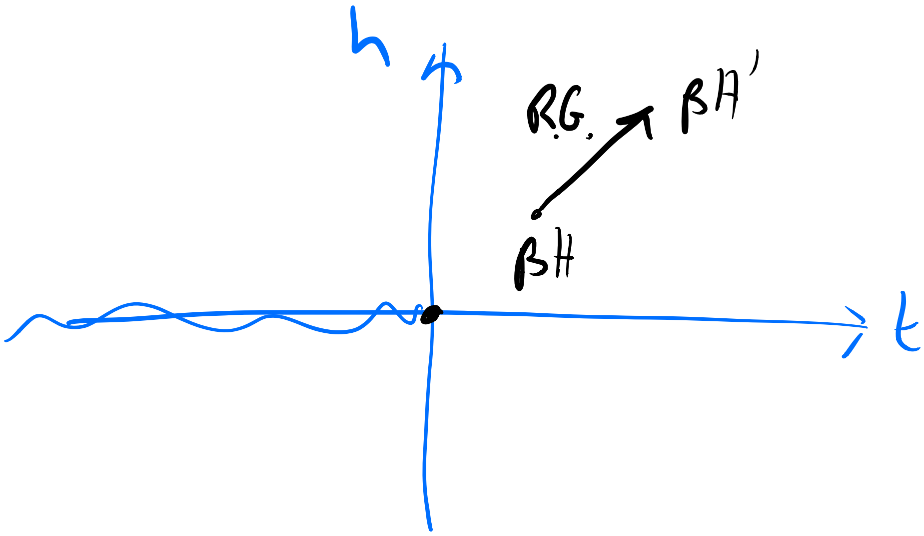
$$W[m] \rightarrow W'[m']$$

<u>Kadanoff (I)</u>

Close to criticality, W' and W are statistically self-similar

- If original βH at $t = h = 0$, there is no characteristic length scale $(\beta H')$ is also at criticality.
- However, if βH is off criticality, then $(\beta H')$ is further away since

$$\xi' = \frac{1}{b}\xi \quad \text{is smaller.}$$



Kadanoff (II)

Mapping is analytic since it only eliminates short length scale fluctuations.

$$\begin{cases} t'(b;t,h) = A(b)t + B(b)h + O(t^2, h^2, th) \\ h'(b;t,h) = C(b)t + D(b)h + O(t^2, h^2, th) \end{cases}$$

By symmetry $C(b) = 0$ (since h would spontaneously break symmetry)

$B(b) = 0$ $\leadsto$ symmetry of $h \to -h$

Commutativity

$$A(b_1 \cdot b_2) = A(b_1)\, A(b_2)$$

$$A(1) = 0$$

$$A(b) = b^{y_t}$$

and $$D(b) = b^{y_h}$$

At lowest order

$$\begin{cases} t_b \equiv t'(b) = b^{y_t}\, t \\ h_b \equiv h'(b) = b^{y_h}\, h . \end{cases}$$

Consequences

1. Free energy

$$Z = \int \mathcal{D}m\, W[\underline{m}] = \int \mathcal{D}\underline{m}'\, W'[m'] = Z'$$

just integrated out but no discarded info.

$$f(t,h) = -\frac{\log Z}{V} = -\frac{\log Z'}{V' b^d} = b^{-d} f(t_b, h_b)$$

function are the same
— no need terms generated by RG.

$$\therefore \; f(t,h) = b^{-d} f(b^{y_t} t, b^{y_h} h)$$

Choose b s.t. $b^{y_t} t = \text{const.}$ (say 1)

$$\Rightarrow \quad b = t^{-\frac{1}{y_t}}$$

$$\Rightarrow f(t,h) = t^{\frac{d}{y_t}} f\left(1, \frac{h}{t^{y_h/y_t}}\right)$$

$$\equiv t^{2-\alpha} g_f\left(\frac{h}{t^\Delta}\right)$$

where $2-\alpha = \frac{d}{y_t}$ and $\Delta = \frac{y_h}{y_t}$

So y_h and y_t determine all critical exponents!

2. Correlation length

$$\xi' = \frac{\xi}{b}$$

$$\xi(t,h) = b\,\xi(b^{y_t}t,\, b^{y_h}h)$$

Set $b^{y_t}t = 1 \Rightarrow \xi(t,h) = t^{-\frac{1}{y_t}}\, g_\xi\left(\frac{h}{t^{y_h/y_t}}\right)$

$$\nu = \frac{1}{y_t}$$

So $(2-\alpha) = \nu d$ a.k.a. Josephson.

3. <u>Magnetisation</u>

$$m = -\frac{\partial f}{\partial h}$$

$$= \frac{1}{V}\frac{\partial \log \mathcal{Z}}{\partial h}$$

$$= \frac{1}{b^d V'}\,\frac{1}{b^{-y_h}}\,\frac{\partial \log \mathcal{Z}'}{\partial h}$$

$$= b^{y_h - d}\, m\left(b^{y_t} t,\, b^{y_h} h\right)$$

$$= t^{\frac{d-y_h}{y_t}}\, g_m\left(\frac{h}{t^{y_h/y_t}}\right)$$

$$\beta = \frac{d-y_h}{y_t}$$