Functional Integrals – a.k.a. field theory

$$Z = \int \mathcal{D}m \, e^{-\beta H[m]}$$

cf. path integral saddle pt $\leftrightarrow$ classical path

fluctuations $\leftrightarrow$ quantum corrections

Gaussian integral in 1 real variable

$$Z_1 = \int_{-\infty}^{\infty} d\varphi \, e^{-\frac{1}{2G}\varphi^2 + h\varphi}$$

$$= \sqrt{2\pi G} \, e^{\frac{1}{2}Gh^2}$$

$$\langle \varphi \rangle = \frac{\partial}{\partial h} \log Z_1 = hG$$

$$\frac{1}{Z_1}\frac{\partial Z_1}{\partial h}$$

Define the cumulant

$$\langle \varphi^r \rangle_c = \left.\frac{\partial^r}{\partial k^r}\right|_{k=0} \log \langle e^{k\varphi} \rangle$$

A student exercise

$$\langle e^{k\varphi}\rangle = \frac{\sqrt{2\pi G}\; e^{\frac{1}{2}G(h+k)^2}}{\sqrt{2\pi G}\; e^{\frac{G}{2}h^2}} = e^{\frac{G}{2}(k^2+2hk)}$$

$$\langle \varphi^2\rangle_c = \partial_k^2\Big|_{k=0} \log\langle e^{k\varphi}\rangle$$

$$= \langle \varphi^2\rangle - \langle\varphi\rangle^2$$

Var $\quad$ 2nd cumulant

$$= \partial_k^2\Big|_{k=0} \frac{G}{2}(k^2+2hk)$$

$$= G$$

N.B. Gaussian ensembles

$$\langle \varphi^r\rangle_c = 0 \qquad r>2.$$

Many real variables

$$Z_N = \int_{-\infty}^{\infty} \prod_{i=1}^{N} d\varphi_i \; e^{-\frac{1}{2}\varphi^T G^{-1}\varphi + \underline{h}\cdot\varphi}$$

real symmetry

$\Rightarrow$ Possible to diagonalize use unitary matrix.

$$u G^{-1} u^{-1} = \hat{G}^{-1}$$

eigenvectors — diagonal matrix of eigenvalues

$$\frac{1}{2} \varphi^T G^{-1} \varphi - \underline{h}\cdot\varphi = \frac{1}{2} \underbrace{\varphi^T u^{-1}}_{\underline{x}^T} \underbrace{u G^{-1} u^{-1}}_{\hat{G}^{-1}} \underbrace{u\varphi}_{\underline{x}} - \underline{h}^T u^{-1} u \varphi$$

$u^{-1} u$

$$= \frac{1}{2} \underline{x}^T \hat{G}^{-1} \underline{x} - \frac{1}{2}\left[\underline{h}^T u^{-1} \underline{x} + \underline{x}^T u \underline{h} \right]$$

Let $\underline{x}^T = \underline{\tilde{x}}^T + \underline{h}^T u^{-1} \hat{G}$

$$\underline{x} = \underline{\tilde{x}} + \hat{G} u \underline{h}$$

$$\frac{1}{2} \varphi^T G^{-1} \varphi - \underline{h}\cdot\varphi = \frac{1}{2}\left[\underline{\tilde{x}}^T + \underline{h}^T u^{-1} \hat{G}\right] \hat{G}^{-1} \left[\underline{\tilde{x}} + \hat{G} u \underline{h}\right]$$
$$- \frac{1}{2}\left[\underline{h}^T u^{-1} (\underline{\tilde{x}} + \hat{G} u \underline{h}) + (\underline{\tilde{x}}^T + \underline{h}^T u^{-1} \hat{G}) u \underline{h} \right]$$

$$= \frac{1}{2} \underline{\tilde{x}}^T \hat{G}^{-1} \underline{\tilde{x}} - \frac{1}{2} \underline{h}^T u^{-1} \hat{G} u \underline{h}$$

$$= \frac{1}{2}\tilde{x}^T \hat{G}^{-1}\tilde{x} - \frac{1}{2}\underline{h}^T G \underline{h}$$

$$Z_N = \int_{-\infty}^{\infty}\prod_i dx_i \; e^{-\frac{1}{2}\underline{x}^T \hat{G}^{-1}\underline{x} + \frac{1}{2}\underline{h}^T G \underline{h}}$$

$$= \sqrt{\det(2\pi G)}\; e^{\frac{1}{2}\underline{h}^T G \underline{h}}$$

Set of decoupled Gaussians $\det = \prod_i \lambda_i$

$$\langle \varphi_i \cdots \varphi_j \rangle_c = \partial_{k_i} \cdots \partial_{k_j}\Big|_{k=0} \log\langle e^{\underline{k}\cdot\underline{\varphi}}\rangle$$

If Gaussian

$$\langle e^{\underline{k}\cdot\underline{\varphi}}\rangle = \exp\left[\frac{1}{2}\underline{k}^T G \underline{k} + \underline{h}^T G \underline{k}\right]$$

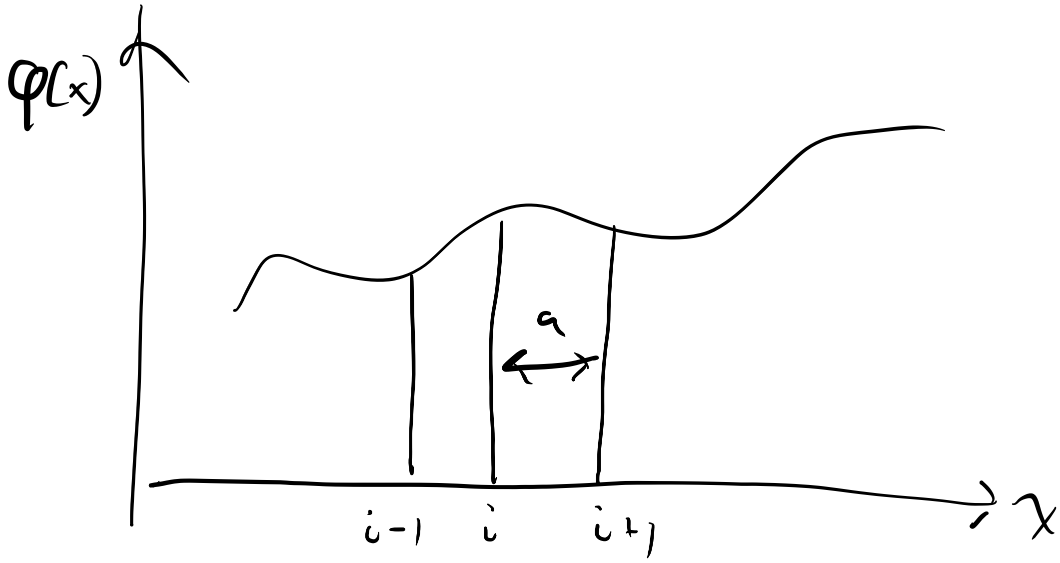
$$\langle \varphi_i \rangle_c = \sum_j G_{ij} h_j = G_{ij} h_j$$

$$\langle \varphi_i \varphi_j \rangle_c = G_{ij}$$

If $A = \underline{a}\cdot\underline{\varphi}$

$$\langle e^{A} \rangle = e^{\langle A \rangle_c + \frac{1}{2}\langle A^2 \rangle_c}$$

Taking the continuum limit $\rightsquigarrow$ Gaussian functional integral.



$N \to \infty$, $a \to 0$

$$\varphi_i \to \varphi(x)$$

labels

$$G^{-1}_{ij} \longrightarrow G^{-1}(\underline{x}, \underline{x}') \equiv \langle x | \hat{G}^{-1} | x' \rangle$$

operator

$$Z = \int \mathcal{D}\varphi(\underline{x}) \exp\left[-\int d^d\underline{x} \int d^d\underline{x}' \, \frac{1}{2} \varphi(x) G^{-1}(\underline{x}, \underline{x}') \varphi(\underline{x}') + \int d^d\underline{x} \, h(\underline{x}) \varphi(\underline{x}) \right]$$

$$\propto (\det G)^{\frac{1}{2}} \exp\left[\frac{1}{2}\int d^d\underline{x} \int d^d\underline{x}' \; h(\underline{x}) G(\underline{x}, \underline{x}') h(\underline{x}')\right]$$

where $\int d^d\underline{x}' \; G^{-1}(\underline{x}, \underline{x}') G(\underline{x}', \underline{x}'') = \delta^d(\underline{x} - \underline{x}'')$

Green's function

"propagator".

<u>Averages</u>

$$\langle \varphi(x) \rangle_c = \int d^d\underline{x}' \; G(\underline{x}, \underline{x}') h(\underline{x}')$$

and $\langle \varphi(\underline{x}) \varphi(\underline{x}') \rangle_c = G(\underline{x}, \underline{x}')$

"connected Green's function"

$$Z = \int \mathcal{D}\varphi(\underline{x}) \exp\left[-\frac{1}{2}\int d^d\underline{x} \left(\frac{\varphi^2}{\xi^2} + (\underline{\nabla}\varphi)^2\right)\right]$$

$-\varphi \nabla^2 \varphi$

$$= \int \mathcal{D}\varphi(\underline{x}) \exp\left[-\frac{1}{2}\int d^d x \int d^d\underline{x}' \; \varphi(\underline{x}) \delta^d(\underline{x} - \underline{x}')\right.$$

$$\underbrace{(-\nabla'^2 + \xi^{-2})}_{G^{-1}(\underline{x}, \underline{x}')} \varphi(\underline{x}') \Big]$$

i.e. $$(-\nabla'^2 + \xi^{-2})\, G(\underline{x}', \underline{x}) = \delta^d(\underline{x}' - \underline{x})$$

Fourier transform

$$G(\underline{q}) = \frac{1}{q^2 + \xi^{-2}}$$

$$\langle \varphi(\underline{q})\, \varphi(\underline{q}') \rangle_c = (2\pi)^d\, \delta^d(\underline{q} + \underline{q}')\, G(\underline{q})$$