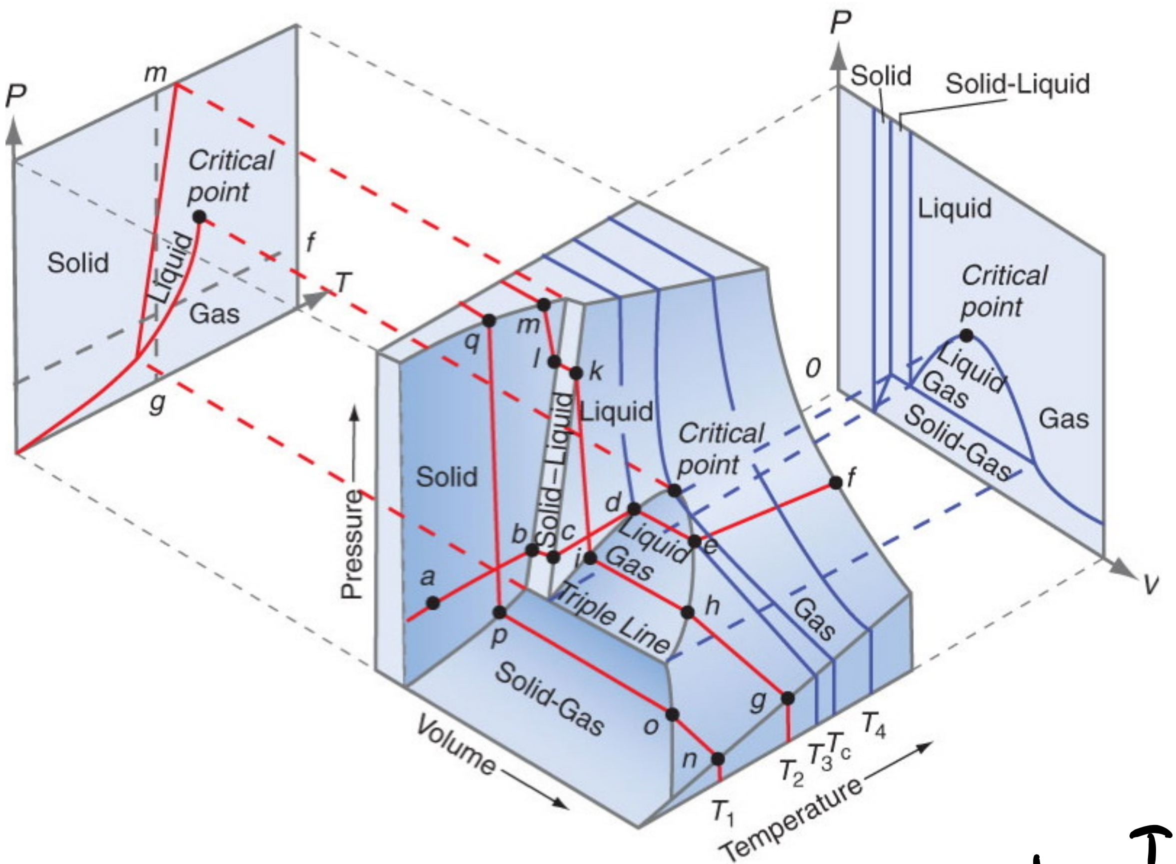
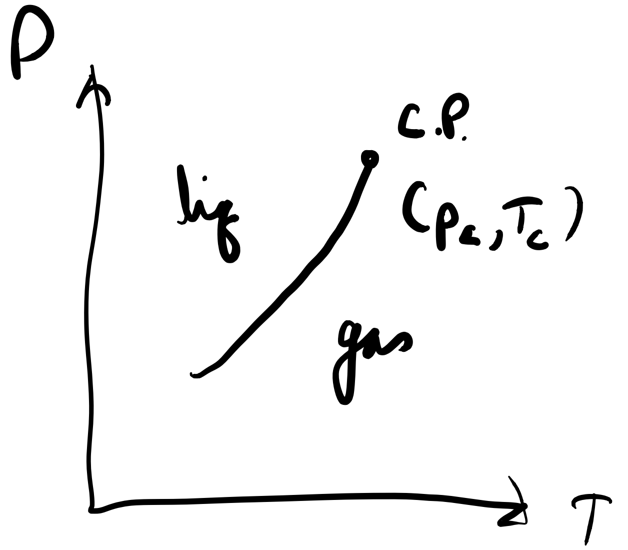
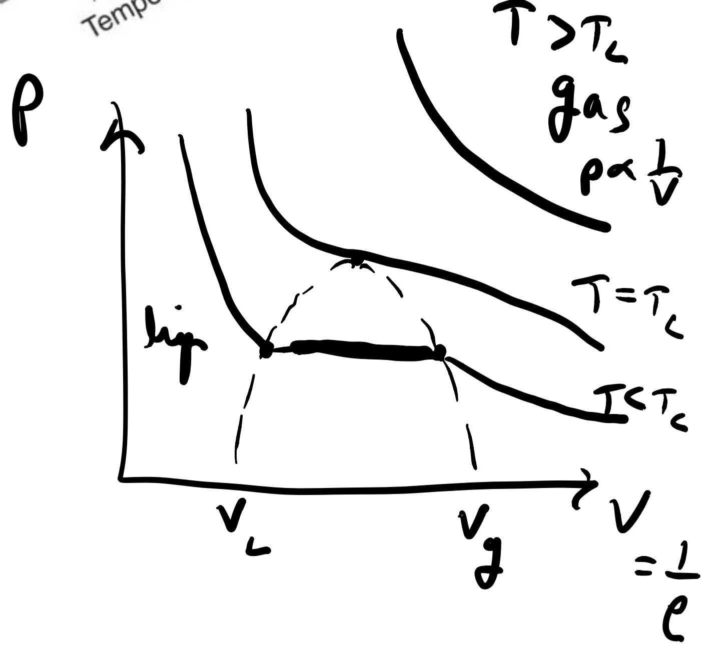
- Anatomy of typical continuous phase transitions
- Motivate statistical field theory

Let us consider the simplest example: liquid-gas



Points to note

1). Coexistence line terminates at a critical point

2). For $T < T_c$, liquid ($\rho_L = \frac{1}{V_L}$) and gas ($\rho_g = \frac{1}{V_g}$) coexist.

3). Liq → gas transition involves a discontinuous change of volume V (except at critical point) where latent heat is exchanged.
(This transition is called first order.)

$\Delta\rho \rightarrow$ order parameter.

4. gas → liquid can occur without a phase transition by going around the critical point.

Close to the critical point

5. As $T \rightarrow T_c^-$ $\rho_L \rightarrow \rho_g$

6. As $T \rightarrow T_c^+$ "isothermal" compressibility

$$K_T = -\frac{1}{V}\frac{\partial V}{\partial p}\Big|_T \longrightarrow \infty$$

$$\searrow \frac{1}{\rho}\frac{\partial \rho}{\partial p}\Big|_T$$

7. System becomes "milky" near critical point
- critical opalescence
cf. boiling kettle, clouds

$\Rightarrow$ characteristic fluctuation at length scale of light.
$\rightarrow$ i.e. $\gg$ larger than typical particle spacing.

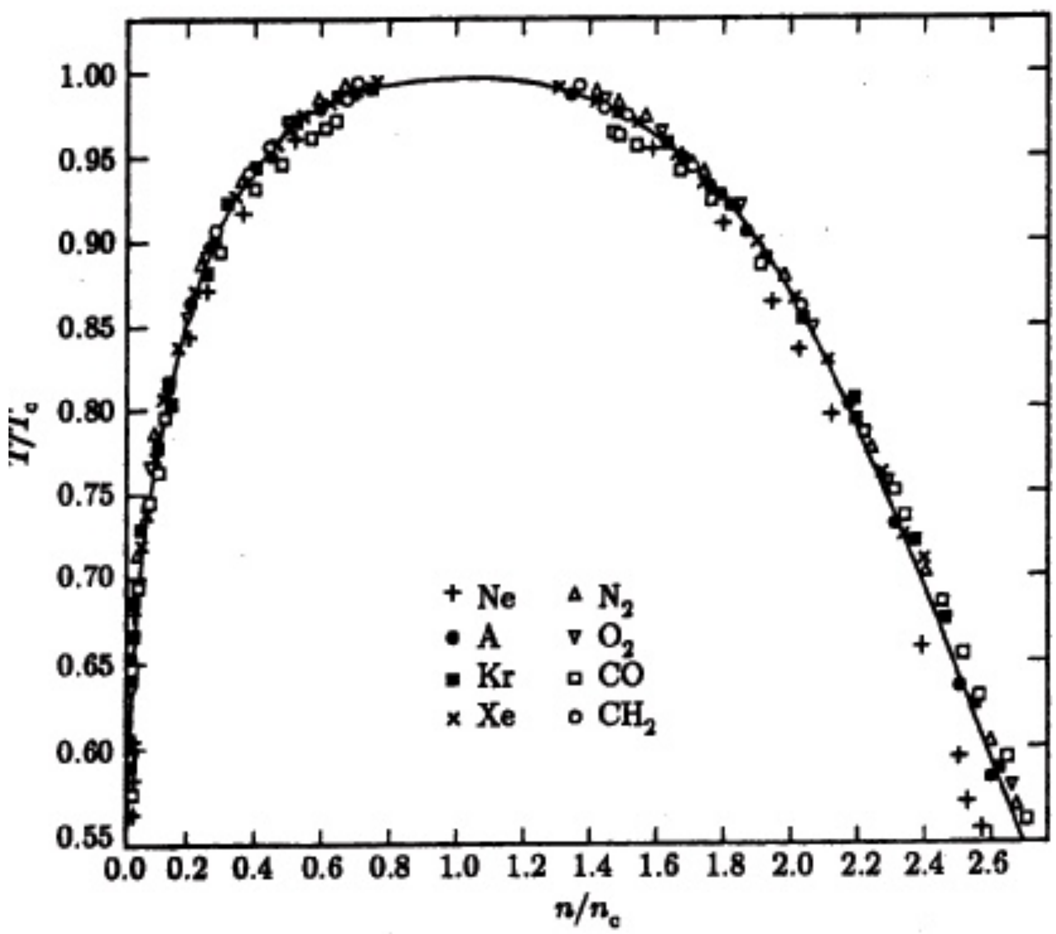


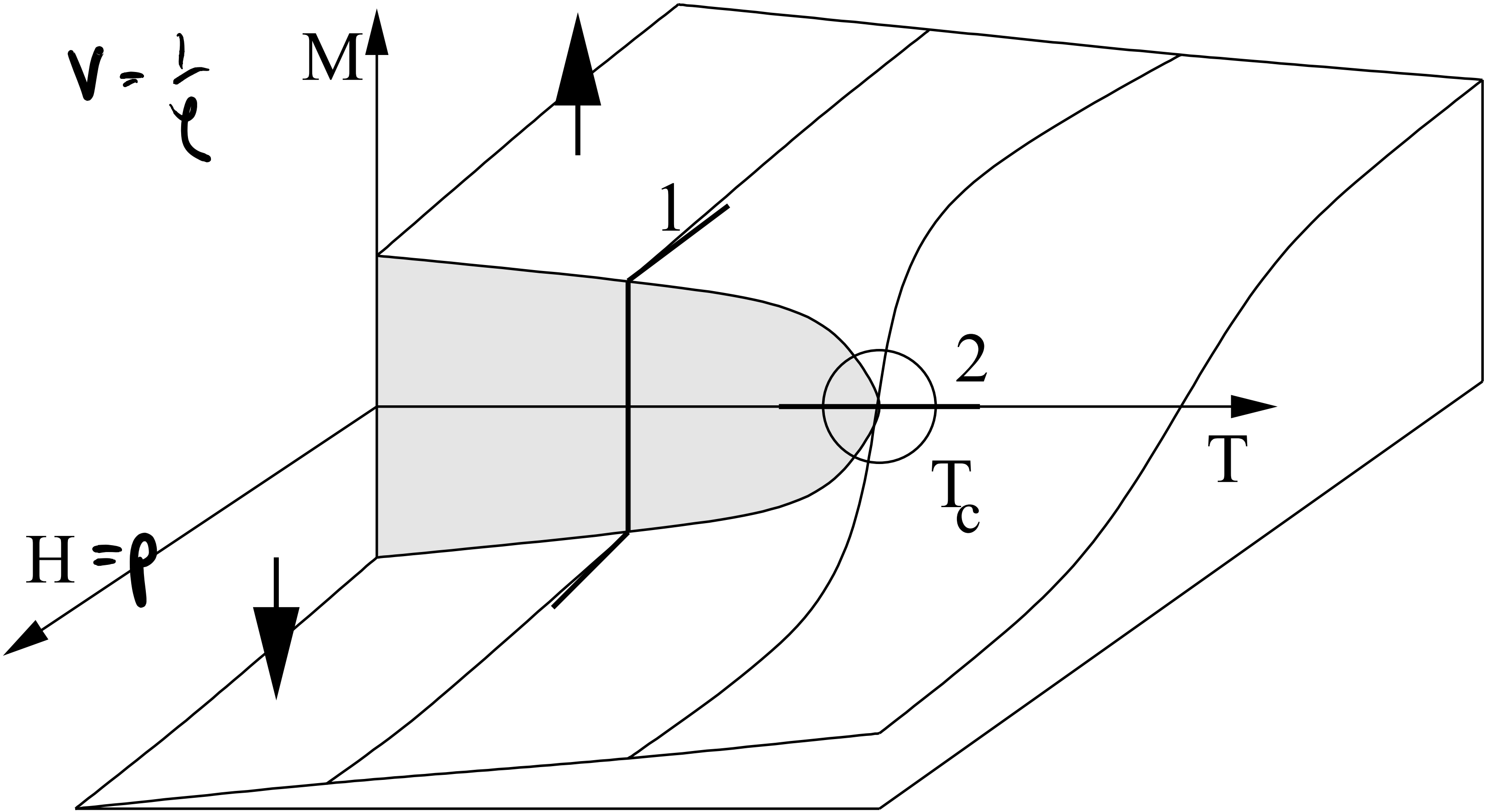

Fig. 4.4.4. Phase boundary in units of reduced temperature and density for eight different molecular fluids near their liquid-gas transitions. Note the universal behavior and the fact that the solid line is $\Delta\phi \propto (T_c - T)^\beta$ with $\beta = 1/3$ rather than the mean-field result $\beta = 1/2$. [E.A. Guggenheim, *J. Chem. Phys.* 13, 253 (1945).]

Ising model — lattice of spins $\pm 1$

$$H = -J \sum_{\langle ij \rangle} \sigma_i \sigma_j$$

nearest neighbours

$$M = \sum_i \sigma_i$$



V = 1/ρ
M
1
2
T
$T_c$
H = ρ

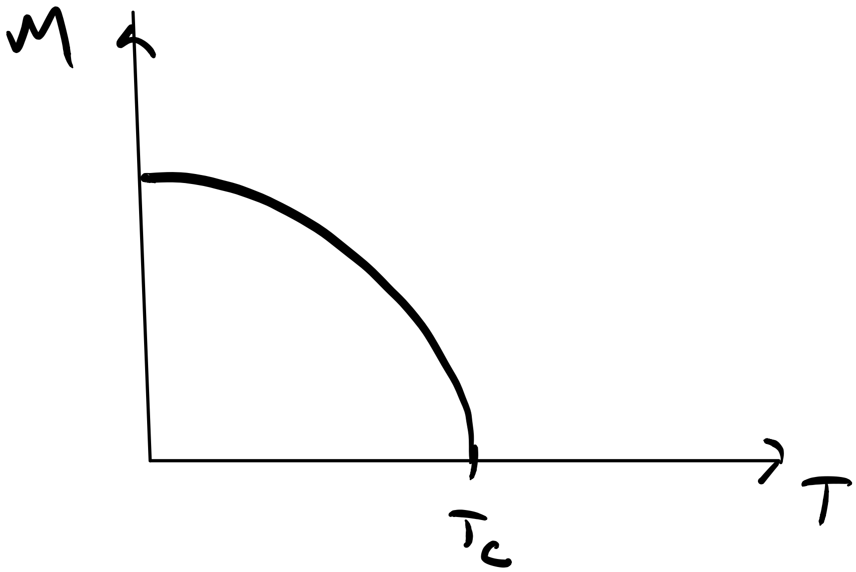
Note that both liquid $\leftrightarrow$ gas and Ising model have identical topologies. (after rotation)

Points to note

As $T \to T_c^-$ (at $H=0$), $M \to 0$

2nd order continuous transition

as in liq $\to$ gas




At $T = T_c$ (and $H=0$), there is spontaneous symmetry breaking, $M \neq 0$

# Critical Phenomena

Special role played by critical point (CP)

On approaching CP, the correlation length diverges

$\Rightarrow$ scale invariance, universality, non-analytic response functions with a set of critical exponents (a.k.a. fingerprint of transition)

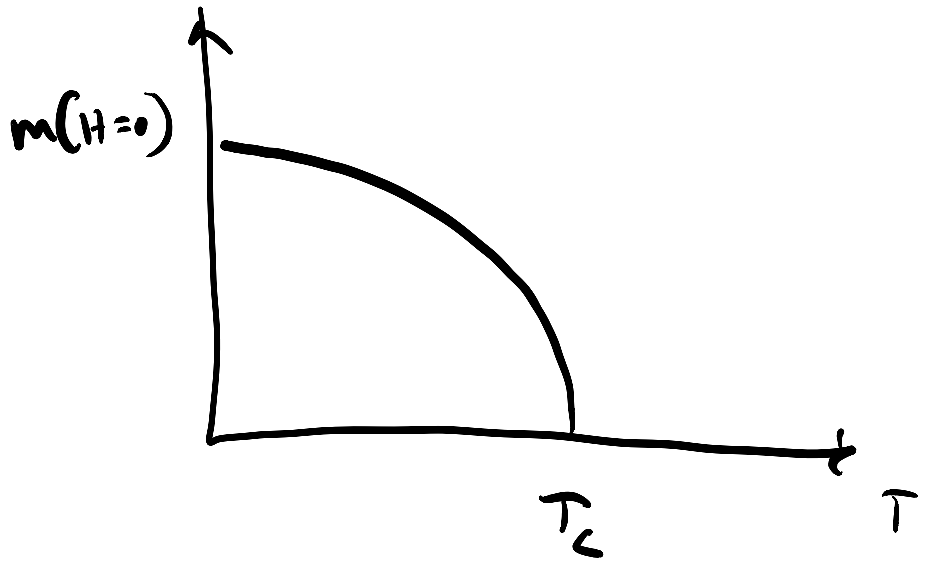
## Definitions

Order parameter

- distinguishes phase on the coexistence line and vanishes at CP

e.g. $M$, $\rho - \rho_c$

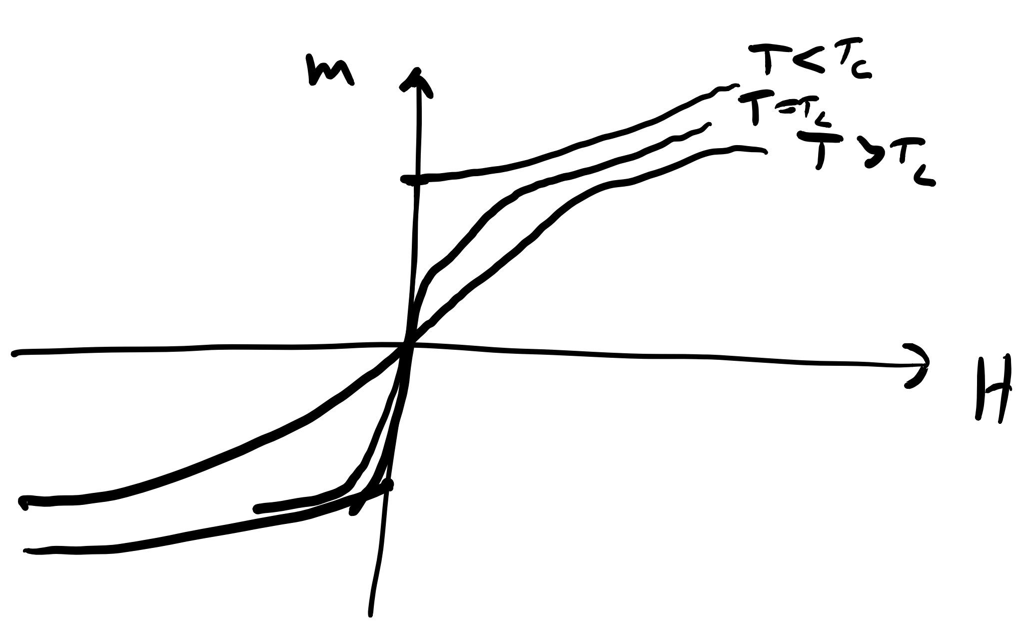
define $m(H,T) = \dfrac{M(H,T)}{V}$




$$m(H=0^+, T) \sim \begin{cases} (-t)^\beta & t<0 \\ 0 & t>0 \end{cases}$$

where $t = \dfrac{T-T_c}{T_c}$ – reduced temperature

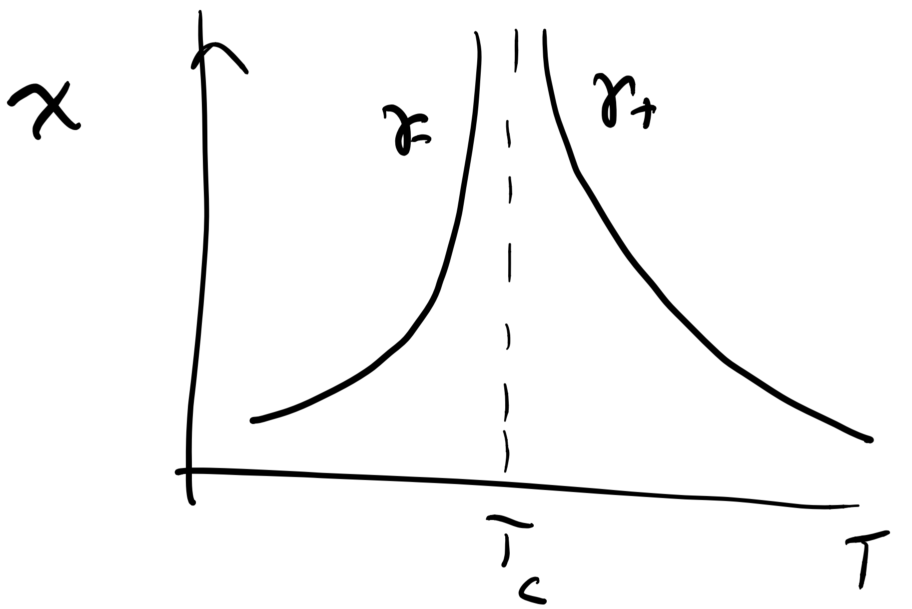
$$\left[ \beta = \lim_{t \to 0^-} \frac{\log|m|}{\log|t|} \right]$$



$$m(H, T=T_c) \sim H^{\frac{1}{\delta}} \qquad H>0$$

Susceptibility $\quad \chi = \dfrac{\partial M}{\partial H}(H,T)$

$$\chi_{\pm} = |t|^{-\gamma_{\pm}}$$

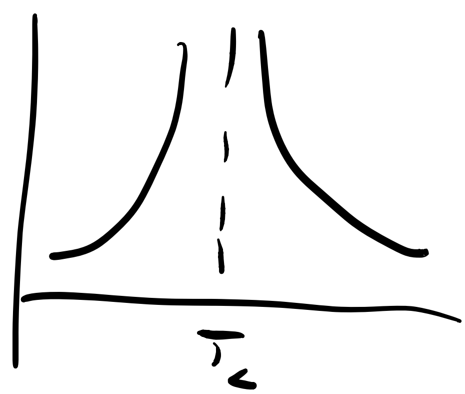



$\gamma_+ = \gamma_- = \gamma$

Heat capacity

$$C_{\pm} = \frac{\partial E}{\partial T} \sim |t|^{-\alpha_{\pm}}$$

$\alpha_+ = \alpha_- = \alpha$



# Correlation functions

divergence of response function $\Leftrightarrow$ long range correlation

$$Z = \operatorname{tr} e^{-\beta(H_0 - hm)}$$

$$\langle M \rangle = \frac{1}{Z} \operatorname{Tr} M e^{-\beta(H_0 - hm)}$$

$$= \frac{\partial \log Z}{\partial(\beta h)}$$

$$\chi = \frac{1}{V} \frac{\partial \langle M \rangle}{\partial h} = \frac{\beta}{V} \frac{\partial \langle m \rangle}{\partial(\beta h)}$$

$$= \frac{\beta}{V} \left( \langle m^2 \rangle - \langle m \rangle^2 \right)$$

$$= \frac{\beta}{V} \operatorname{Var} m$$

$$\Leftrightarrow \quad V k_B T \chi = \langle m^2 \rangle - \langle m \rangle^2$$

$$M = \int d^d \underline{r}\; m(\underline{r})$$

$$V k_B T \chi = \int d^d \underline{r} \int d^d \underline{r}' \Big[ \langle m(\underline{r})\, m(\underline{r}') \rangle - \langle m(\underline{r}) \rangle \langle m(\underline{r}') \rangle \Big]$$

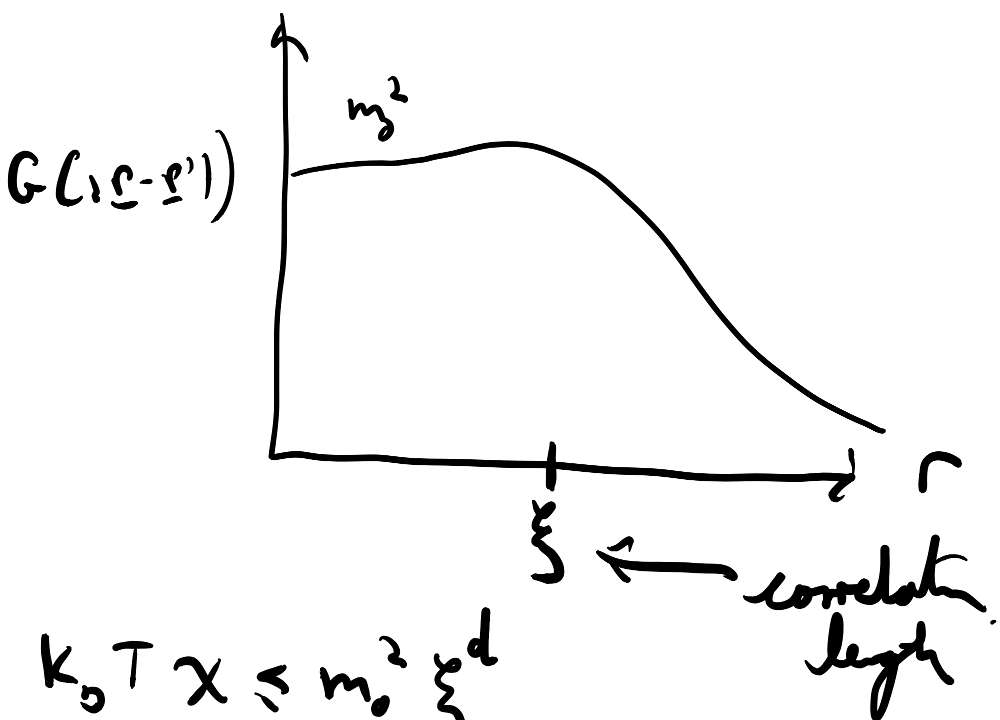
$$\equiv$$

$$G(\underline{r} - \underline{r}') = \langle m(\underline{r})\, m(\underline{r}') \rangle_c$$

$\uparrow$

Connected correlation function.

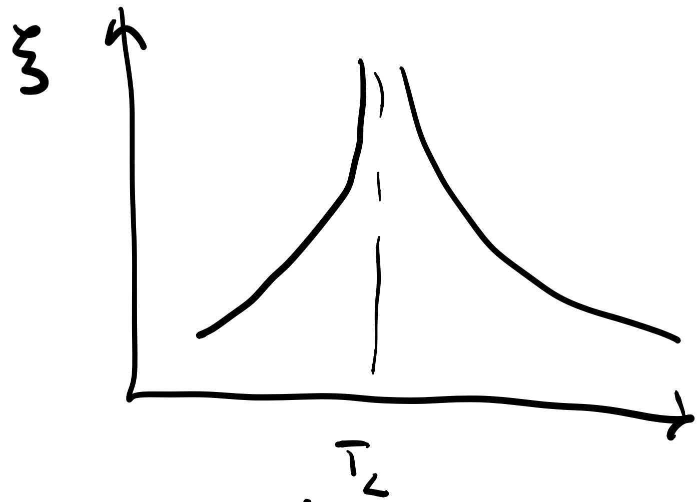
$$k_B T \chi = \int d^d \underline{r}\; \langle m(\underline{r})\, m(\underline{0}) \rangle_c$$




$$k_B T \chi \lesssim m_0^2\, \xi^d$$

If $\chi \to \infty$ at $T_c$

then $\xi \to \infty$ at $T_c$.



$\xi_\pm \sim |t|^{-\nu_\pm}$