Quantum Phase Transitions

- Non-analytic behavior of ground state of the infinite system — i.e. zero temperature phase transition

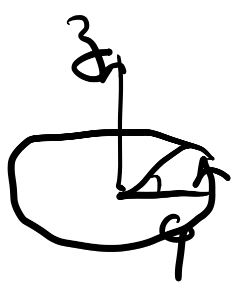
O(2) rotor example

Used to describe superconducting islands coupled by Josephson junctions.

$$\hat{H} = \sum_i \frac{L_i^2}{2m} - g \sum_{\langle ij \rangle} \hat{\underline{x}}_i \cdot \hat{\underline{x}}_j$$

charging energy

Josephson coupling



$$\hat{H} = -\frac{1}{2m} \sum_i \frac{\partial^2}{\partial \varphi_i^2} - g \sum_{\langle ij \rangle} \cos(\varphi_i - \varphi_j)$$

$g > 0$

Qualitative analysis of G.S.

$$mg \ll 1 \iff m \ll \frac{1}{g}$$

- first term / K.E. dominates

$\therefore$ energy eigenvalues are approximately the angular momentum eigenstate

$$\psi(\varphi_i) = \prod_i e^{i m_\ell^i \varphi_i}$$

$$m_\ell \in \mathbb{Z}$$

Ground state is when $m_\ell^i = 0 \ \forall i$

$$\Rightarrow \langle H \rangle = 0$$

$\langle \psi | \cos(\varphi_i - \varphi_j) | \psi \rangle = 0$ i.e. no L.R.O.

Second term can be treated as a perturbation in mg mixing the degenerate states

$\therefore$ expect energy gap.

$mg \gg 1$ - second term dominates

Since $g > 0$, rotor alignment to be favoured. - order quantum phase.

Energy eigenstates are φ eigenstates approximately

$$\psi(\varphi) = \prod_i \delta(\varphi_i - c_i)$$

The ground state is has its symmetry spontaneously broken and w.l.o.g. we may set $c_i = 0 \ \forall i$.

$$\langle \psi | \cos(\varphi_i - \varphi_j) | \psi \rangle = 1$$

$\Rightarrow$ L.R.O.

Do fluctuations destroy L.R.O.?

- We expect a <u>gapless</u> energy dispersion
- Goldstone mode.

$\therefore$ 2 regimes have qualitatively different behaviour (gapped vs gapless)

$\therefore$ $T=0$ phase transition

$\Rightarrow$ need to analyze the stability of the phases.

O(2) quantum - classical mapping

$$H = \sum_i \frac{L_i^2}{2m} - g \sum_{\langle ij \rangle} \hat{\underline{x}}_i \cdot \hat{\underline{x}}_j$$

$$Z = \int \mathcal{D}\varphi_i(\tau)\, e^{-\beta H[\varphi_i(\tau)]}$$

$$\varphi_i(\beta) - \varphi_i(0) = 2\pi n$$

$$\beta H[\varphi_i(\tau)] = \int_0^\beta d\tau \sum_{i=1}^{N} \frac{m}{2}(\partial_\tau \varphi_i)^2 - g \sum_{\langle ij \rangle} \cos(\varphi_i - \varphi_j)$$

At zero temperature ($\beta \to \infty$), the d-dimensional quantum Hamiltonian maps to a $d+1$ dimensional classical system, - with 1 Goldstone mode.

By Mermin-Wagner theorem, we expect no L.R.O. if $d \leq 1$.

Check by expanding the order state and find $\langle \varphi_i^2 \rangle$.

$$\beta H = \int_{-\infty}^{\infty} d\tau\, \frac{m}{2} \sum \left(\frac{\partial \varphi_i}{\partial \tau}\right)^2 + \frac{g}{2} \sum_{\langle ij \rangle} (\varphi_i - \varphi_j)^2$$

$$\varphi_i = \sum_q{}' \varphi_q e^{i \underline{x}_i \cdot \underline{q}}$$

$$\beta H = \int_{-\infty}^{\infty} d\tau \sum_q \left[\frac{m}{2}\left|\frac{\partial \varphi_q}{\partial \tau}\right|^2 + \frac{g}{2} q^2 |\varphi_q|^2\right]$$

$$\varphi_q(\tau) = \int_{-\infty}^{\infty} \frac{d\omega}{(2\pi)} \varphi_q(\omega) e^{i\omega t}$$

$$\rightsquigarrow \quad \beta H = \int_{-\infty}^{\infty} d\omega \sum_{q \in B.Z.}{}' \frac{1}{2}\left[\frac{m\omega^2}{\cancel{(2\pi)}} + g q^2\right] |\varphi_q(\omega)|^2$$

S.H.O.

$$\langle \varphi_i^2(\tau=0) \rangle = \int d\omega \sum_{q \in B.Z.}{}' \frac{1}{m\omega^2 + g q^2}$$

$$= \frac{1}{\sqrt{mg}} \int \frac{d^d q}{(2\pi)^d} \frac{1}{|q|} \int d\omega \frac{1}{1+\omega^2}$$

$$\propto \int \frac{d^d q}{(2\pi)^d} \frac{1}{|q|}$$ which diverges if $d \leq 1$

Excitation spectrum $\omega = \sqrt{\frac{g}{m}}\,|q|$

$$\mathcal{Z} \hat{=} \mathcal{N} \mathcal{Z}_\varphi \int \mathcal{D}(\psi_i, \psi_i^*)\, e^{-S[\psi(\underline{r},\tau)]}$$

$$S[\psi(\underline{r},\tau)] = \int \frac{d^d r}{a^d}\, d\tau \Big\{ t|\psi|^2 + \frac{a^2}{2gd^2}|\underline{\nabla}\psi|^2$$

$$+ 8m^3|\partial_\tau \psi|^2 + 28m^3|\psi|^4 \Big\}.$$

$$t = \frac{1}{2gd} - 4m$$

$t=0$ - zero T phase transition.

$$g = \frac{1}{8md}$$

By rescaling $\left.\begin{array}{l} \underline{r} \to \frac{a}{\sqrt{2}\,g}\underline{r} \\ \tau \to m^{\frac{3}{2}}\tau \end{array}\right] \rightsquigarrow S$ is isotropic

$$\Rightarrow \quad \xi \sim \xi_\tau \sim \frac{1}{|t|^\nu}$$

Generally S is <u>not</u> isotropic

$$\xi \sim \frac{1}{|t|^\nu}$$

$$\xi_\tau \sim \frac{1}{|t|^{z\nu}}$$

z – dynamical exponent.