Problem Set III

Lent 2023

Quantum Phase Transitions

1. Following from the question about the Lifshitz point in Problem Set I, we are now going to apply the RG scaling to find the upper critical dimension of several quantum phase transitions. The correspondence is most easily seen if we let $x_\parallel \to \tau$, $\mathbf{x}_\perp \to \mathbf{x}$, and $q_\parallel \to \omega$, $\mathbf{q}_\perp \to \mathbf{k}$. Note that unlike GL Hamiltonians for classical systems, GL Hamiltonians (better known as imaginary time actions in this context) for quantum systems are in general complex.

   The following effective Hamiltonian is realised in some $(d-1)$-dimensional quantum systems

$$\begin{aligned}\beta H &= \int d\tau\, d^{d-1}\mathbf{x} \int d\tau'\, d^{d-1}\mathbf{x}'\, \Psi(\mathbf{x},\tau) K(\mathbf{x}-\mathbf{x}', \tau-\tau')\Psi(\mathbf{x}',\tau') \\ &\quad + \int d\tau\, d^{d-1}\mathbf{x} \left(\frac{t}{2}\Psi^2(\mathbf{x},\tau) + u\Psi^4(\mathbf{x},\tau)\right), \qquad (1)\end{aligned}$$

   where $\Psi(\mathbf{x},\tau)$ has N real components and the Fourier transform of the kernel $K(\mathbf{x},\tau)$ for small frequency is given below. The kernel is translationally invariant along $\mathbf{x}$ and τ. For each kernel, find the upper critical dimension d_u for validity of the Gaussian exponents. [*Hint: Rescale $\omega' = c\omega$ and $\mathbf{k}' = b\mathbf{k}$ and choose an appropriate renormalisation factor z for the order parameter so that the couplings that govern the long distance behaviour along τ and $\mathbf{x}$ both remain the same.*]

   (a) $K(\omega, k) = d\omega + ek^2$ ($T = 0$ antiferromagnetic transition for itinerant electrons when $N = 3$),

   (b) $K(\omega, k) = id\omega + ek^2$ (superfluid transition when $N = 2$),

   (c) $K(\omega, k) = d|\omega|/k + ek^2$ (ferromagnetic transition for itinerant electrons when $N = 3$),

   (d) $K(\omega, k) = d\omega^2 + ek^2$ (superfluid transition for bosons on a lattice and at a commensurate density when $N = 2$),

   (e) $K(\omega, k) = d|\omega| + e\omega^2 + fk^2$ (superconducting transition for dirty d-wave superconductor when $N = 2$).

2. *Quantum-Classical Mapping:* The following problem explores the quantum-classical analogy in the context of O(N) quantum rotor systems.

   In $d = 3$ the quantum O(2) rotor Hamiltonian is given by

$$\hat{H}_{\mathrm{O}(2)} = \sum_i \frac{\hat{L}_i^2}{2m} - \frac{g}{2}\sum_{<ij>} \cos(\hat{\theta}_i - \hat{\theta}_j), \qquad (2)$$

where $\hat{L}_i = -i\frac{\partial}{\partial\theta}$ is the angular momentum operator and $\hat{\theta}_i$ the conjugate azimuthal angle operator, $[\hat{\theta}, \hat{L}_j] = i\delta_{ij}$. Close to the critical point the zero-temperature parition function of the Hamiltonian is well described by the following Ginzburg-Landau action

$$S[\Psi(\mathbf{r},\tau)] = \int \frac{d^3\mathbf{r}}{a^3} d\tau \Big[t|\Psi(\mathbf{r},\tau)|^2 + \frac{a^2 g^{-1}}{18}|\nabla\Psi(\mathbf{r},\tau)|^2 + 8m^3|\partial_\tau\Psi(\mathbf{r},\tau)|^2 + cm^3|\Psi(\mathbf{r},\tau)|^4 \Big],$$

where $\Psi(\mathbf{r},\tau)$ is a complex order parameter.

(a) Outline the derivation of the Ginzburg-Landau action and find the reduced temperature t. In particular, state which angular momentum states participate in the GL action and why, and explain the origin of the quartic term and why $c > 0$.

(b) Write down the quantum field Hamiltonian whose Feynman-path representation generates the action S.

(c) Assuming the validity of the mean-field approximation at the upper critical dimension, find the singular behaviour of the correlation length in the GL action S, close the critical point. Using this result find the corresponding critical behaviour of the energy gap in $\hat{H}_{\mathrm{O}(2)}$.

(d) Now, consider the Hamiltonian $\hat{H}$ in $d = 2$, with the rotors placed on sites of a square lattice. By looking at the stability of high $mg \gg 1$ and low $mg \ll 1$ phases, and deriving the low-energy spectrum in each case, deduce the presence of a zero-temperature phase transition. Identify the universality class to which this transition belongs. By quoting the relevant exponent from the lecture notes, state the singular behaviour of the energy gap at the transition. Finally, comment on the non-zero temperature behaviour of this model.

(e) We now turn to analysisng the $mg = 0$ limit, where the quantum rotors decouple and the second term in the Hamiltonian can be neglected.

Setting $m = 1$, show that the quantum partition function for a single rotor is given by

$$\mathcal{Z} = \sum_{n=-\infty}^{\infty} \exp\left[-\beta\frac{n^2}{2}\right]. \tag{3}$$

Hence, find the low and high temperature limits of the following imaginary time correlator

$$\langle e^{i\hat{\theta}(t)-i\hat{\theta}(0)}\rangle_T = \frac{1}{\mathcal{Z}}\sum_n \langle n|e^{-\beta\hat{H}}e^{i\hat{\theta}(\tau)}e^{-i\hat{\theta}(0)}|n\rangle, \tag{4}$$

where $e^{i\hat{\theta}(\tau)} = e^{\tau\hat{H}}e^{i\hat{\theta}}e^{-\tau\hat{H}}$, $\tau > 0$ and $|n\rangle$ are the energy eigenstates.

(f) Cast as a Feynman path integral, show that the quantum partition function takes the form

$$\mathcal{Z} = \sum_{n=-\infty}^{\infty} \int_{\theta(\beta)=\theta(0)+2\pi n} D\theta(\tau) \exp\left[-\frac{1}{2}\int_0^\beta d\tau \dot{\theta}^2\right],$$

where $\beta = 1/kT$.

(g) Parametrising paths as $\theta(\tau) = 2\pi n\tau/\beta + \theta_p(\tau)$, where $\theta_p(\tau)$ is a periodic function of Euclidean time, show that

$$\mathcal{Z} = \sum_{n=-\infty}^{\infty} \exp\left[-\frac{1}{2}\frac{(2\pi n)^2}{\beta}\right] D\theta(\tau) \exp\left[-\frac{1}{2}\int_0^\beta d\tau\, \dot{\theta_p}^2\right]. \tag{5}$$

Use the Feynman path integral to find the the $T \to 0$ limit of the correlator

$$\langle e^{i\hat{\theta}(\tau)} e^{-i\hat{\theta}(0)} \rangle. \tag{6}$$

How does your answer compare with the result from part (e)? Explain.

(h) By analytically continuing the high-temperature limit of the correlator found in (e) to real time $\tau \to it$, show that

$$\langle e^{i\hat{\theta}(t) - i\hat{\theta}(0)} \rangle_T = e^{-it - Tt^2} \tag{7}$$

and comment on the implications of your result for the relevance of temperature to real time correlators.

The following questions consider finite-size scaling effects around the quantum critical point associated with non-zero temperature, which is a non-examinable topic:

†(i) Consider the behaviour of the 3-dimensional O(2) quantum rotor Hamiltonian at non-zero temperature T. Sketch the RG flow in the $\{t, T\}$ plane, paying particular attention to the phase boundary and its behaviour with t close to the critical point. Identify the classical and quantum critical regions and how the crossover temperatures vary with t. [*You can assume mean-field critical exponents at the upper critical dimension.*]

† (j) How does the susceptibility of the 3-dimensional *quantum* O(2) rotor model diverge in the quantum critical region as $T \to 0$? [*You can assume mean-field critical exponents at the upper critical dimension.*]

† (k) Using the fluctuation-dissipation theorem and your answer to part (e), find the singular behaviour of the correlation length as $T \to 0$ in the quantum critical region.