Problem Set I

Lent 2023

Questions on Ginzburg-Landau Theory

Discontinuous Transitions: In lectures we focussed on the study of Landau theory of second order phase transitions in which the order parameter goes to zero continuously. When the order parameter vanishes discontinuously, the transition is said to be first order. Amongst those first order transitions most commonly encountered in Landau theory there includes the following model:

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1. *Tricritical Point*: In class we examined the Ginzburg-Landau Hamiltonian

$$\beta H = \int d\mathbf{x} \left[\frac{t}{2}\mathbf{m}^2 + u\mathbf{m}^4 + v\mathbf{m}^6 + \frac{K}{2}(\nabla \mathbf{m})^2 - \mathbf{h}\cdot\mathbf{m} \right],$$

with $u > 0$ and $v = 0$. If $u < 0$, then a positive v is necessary to ensure stability.

(a) By sketching the free energy $F(m)$ for various values of t, show that there is a first order transition for $u < 0$ and $h = 0$.

(b) Calculate $\bar{t}$ and the discontinuity in $\bar{m}$ at the transition.

(c) For $h = 0$ and $v > 0$ plot the phase boundary in the (u, t) plane, identifying the phases, and the order of the phase transitions.

(d) The special point $u = t = 0$, separating first and second order phase boundaries, is called a tricritical point. For $u = 0$ calculate the exponents α, β, γ, and δ. (A discussion of the tricritical point and multi-critical points in general can be found on p. 172 in Chaikin and Lubensky — although you should attempt to complete the question yourself before resorting to the text!)

[Recall: $C \sim t^{-\alpha}$; $\bar{m} \sim t^{\beta}$; $\chi \sim t^{-\gamma}$; and $\bar{m} \sim h^{1/\delta}$.]

Fluctuations: The final set of questions on the problem set are concerned with studying the fluctuation corrections to the mean-field.

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2. Following on from the previous question, taking the Ginzburg-Landau Hamiltonian from above *with* $u = 0$:

(a) Calculate the heat capacity singularity as $t \to 0$ using the saddle-point approximation.

(b) Setting

$$\mathbf{m}(\mathbf{x}) = (\bar{m} + \phi_l(\mathbf{x}))\hat{\mathbf{e}}_l + \sum_{\alpha=2}^{n} \phi_t^\alpha(\mathbf{x})\hat{\mathbf{e}}_\alpha,$$

expand βH to quadratic order in longitudinal and transverse fluctuations ϕ.

(c) Following the Fourier analysis developed in the lectures, and making use of the integral identity below, obtain an estimate for the longitudinal and transverse correlation functions $\langle \phi_{l,t}(\mathbf{x})\phi_{l,t}(0)\rangle$.

(d) Taking into account the leading contribution from fluctuations obtain the first correction to the saddle-point free energy.

(e) From this result, obtain the leading fluctuation corrections to the heat capacity.

(f) By comparing the results from parts (a) and (e) obtain the Ginzburg criterion for mean-field theory to apply, and show that, for the tricritical point, the upper critical dimension $d_u = 3$.

(g) A generalised multi-critical point is described by replacing the term vm^6 with $u_{2n}m^{2n}$. Using only power counting arguments, show that the upper critical dimension of the multi-critical point is $d_u = 2n/(n-1)$.

$$\left[-\int \frac{d\mathbf{q}}{(2\pi)^d} \frac{e^{i\mathbf{q}\cdot\mathbf{x}}}{\mathbf{q}^2 + \xi^{-2}} \simeq \begin{cases} \dfrac{|\mathbf{x}|^{2-d}}{(2-d)S_d} & |\mathbf{x}| \ll \xi, \\ \dfrac{\xi^{(3-d)/2}}{(2-d)S_d|\mathbf{x}|^{(d-1)/2}} \exp(-|\mathbf{x}|/\xi) & |\mathbf{x}| \gg \xi. \end{cases} \right]$$

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3. *Spin Waves*: In the XY-model of magnetism, a unit two-component vector $\mathbf{S} = (S_x, S_y)$ (with $\mathbf{S}^2 = 1$), is placed on each site of a d-dimensional lattice. There is an interaction that tends to keep nearest neighbours parallel, i.e. a Hamiltonian

$$\beta H = -K \sum_{\langle ij \rangle} \mathbf{S}_i \cdot \mathbf{S}_j.$$

[The notation $\sum_{\langle ij \rangle}$ is conventionally used to indicate a sum over all *nearest neighbour* pairs (i, j).]

(a) Rewrite the partition function

$$\mathcal{Z} = \int \prod_i d\mathbf{S}_i \exp[-\beta H]$$

as an integral over the set of angles $\{\theta_i\}$ between the spins $\{\mathbf{S}_i\}$ and some arbitrary axis.

(b) At low temperatures (i.e. $K \gg 1$), the angles $\{\theta_i\}$ vary slowly from site to site. In this case expand βH to obtain a quadratic expansion in $\{\theta_i\}$.

(c) For $d = 1$ consider L sites with periodic boundary conditions (i.e. forming a chain). Find the normal (spin-wave) modes $\theta_{\mathbf{q}}$ that diagonalise the quadratic form (by Fourier transformation), and show that the corresponding eigenvalue spectrum, the *dispersion relation*, is given by $K(q) = 2K(1-\cos q)$. [Hint: Compare this result to the phonon dispersion curve obtained from a weakly coupled chain of oscillators — the phonon modes of a lattice.]

(d) Generalise the results from (c) to a d-dimensional simple cubic lattice with periodic boundary conditions.

(e) Obtain an estimate of the contribution of these modes to the free energy in the form of an integral. (Evaluate the classical partition function, i.e. do not quantise the modes.) Without performing the integration explicitly (i.e. by examining the temperature dependence alone), determine the contribution of these modes to the specific heat. Show that, at high temperatures, this result is in accord with the equipartition theorem.

(f) Find an expression for $\langle \mathbf{S}_0 \cdot \mathbf{S}_x \rangle = \mathrm{Re}\, \langle \exp[i(\theta_{\mathbf{x}} - \theta_0)] \rangle$ in the form of an integral. Convince yourself that for $|\mathbf{x}| \to \infty$, only $\mathbf{q} \to 0$ modes contribute appreciably to this expression, and hence obtain an expression for the asymptotic limit in dimensions 1, 2, and 3. [Hint: Note that this calculation is simply the discrete analogue of the continuum calculation made in the text.]

Problems on Scaling and the Renormalisation Group

The problems below are designed to develop some of the ideas concerning scaling and the renormalisation group introduced in lectures.

Although it was not discussed in lectures, the first (and in some sense the easiest) application of the Renormalisation Group was to a lattice spin Hamiltonian. The first problem in this section goes, step by step, through the RG transformation for the one-dimensional Ising model (where in fact the RG is exact). If you get lost refer to e.g. Chaikin and Lubensky, section 5.6 p. 242.

1. *The Migdal-Kadanoff Method*: The partition function for the one-dimensional ferromagnetic Ising model with nearest neighbour interaction is given by

$$\mathcal{Z} = \sum_{\{\sigma_i = \pm 1\}} e^{-\beta H[\sigma_i]}, \qquad \beta H = -\sum_{\langle ij \rangle} \left[J\sigma_i\sigma_j + \frac{h}{2}(\sigma_i + \sigma_j) + g \right],$$

   where $\langle ij \rangle$ denotes the sum over neighbouring lattice sites. The Migdal-Kadanoff scheme involves an RG procedure which, by eliminating a certain fraction of the spins from the partition sum, reduces the number of degrees of freedom by a factor of b. Their removal induces an effective interaction of the remaining spins which renormalises the coefficients in the effective Hamiltonian. The precise choice of transformation is guided by the simplicity of the resulting RG. For $b = 2$ (known as *decimation*) a natural choice is to eliminate (say) the even numbered spins.

(a) By applying this procedure, show that the partition function is determined by a renormalised Hamiltonian involving spins at odd numbered sites σ'_i,

$$\mathcal{Z} = \sum_{\{\sigma'_i=\pm 1\}} e^{-\beta H'[\sigma'_i]},$$

where the Hamiltonian $\beta H'$ has the same form as the original but with renormalised interactions determined by the equation

$$\begin{aligned}
&\exp\left[J'\sigma'_1\sigma'_2 + \frac{h'}{2}(\sigma'_1+\sigma'_2) + g'\right] \\
&\quad = \sum_{s=\pm 1} \exp\left[Js(\sigma'_1+\sigma'_2) + \frac{h}{2}(\sigma'_1+\sigma'_2) + hs + 2g\right].
\end{aligned}$$

(b) Substituting different values for σ'_1 and σ'_2 obtain the relationship between the renormalised coefficients and the original. Show that the recursion relations take the general form

$$\begin{aligned}
g' &= 2g + \delta g(J,h), \\
h' &= h + \delta h(J,h), \\
J' &= J'(J,h).
\end{aligned}$$

(c) For $h = 0$ check that no term h' is generated from the renormalisation (as is clear from symmetry). In this case, show that

$$J' = \frac{1}{2}\ln\cosh(2J).$$

Show that this implies a stable (infinite T) fixed point at $J = 0$ and an unstable (zero T) fixed point at $J = \infty$. Any finite interaction renormalises to zero indicating that the one-dimensional chain is always disordered at sufficiently long length scales.

(d) Linearising (in the exponentials) the recursion relations around the unstable fixed point, show that

$$e^{-J'} = \sqrt{2}e^{-J}, \qquad h' = 2h.$$

(e) Regarding e^{-J} and h as scaling fields, show that in the vicinity of the fixed point the correlation length satisfies the homogeneous form ($b = 2$)

$$\begin{aligned}
\xi(e^{-J},h) &= 2\xi(\sqrt{2}e^{-J},2h) \\
&= 2^\ell\xi(2^{\ell/2}e^{-J},2^\ell h).
\end{aligned}$$

Note that choosing $2^{\ell/2}e^{-J} = 1$ we obtain the scaling form

$$\xi(e^{-J}, h) = e^{2J} g_\xi(he^{2J}).$$

The correlation length diverges on approaching $T = 0$ for $h = 0$. However, its divergence is not a power law of temperature. Thus there is an ambiguity in identifying the exponent ν related to the choice of measure in the vicinity of $T = 0$ ($1/J$ or e^{-J}). The hyperscaling assumption states that the singular part of the free energy in d-dimensions is proportional to ξ^{-d}. Hence we expect

$$f_{\text{sing.}}(J, h) \propto \xi^{-1} = e^{-2J} g_f(he^{2J}).$$

At zero field, the magnetisation is always zero, while the susceptibility behaves as

$$\chi(J) \sim \left.\frac{\partial^2 f}{\partial h^2}\right|_{h=0} \sim e^{2J}.$$

On approaching $T = 0$, the divergence of the susceptibility is proportional to that of the correlation length. Using the general form $\langle \sigma_i \sigma_{i+x} \rangle \sim e^{-x/\xi}/x^{d-2+\eta}$ and $\chi \sim \int dx \langle \sigma_0 \sigma_x \rangle_c \sim \xi^{2-\eta}$ we conclude $\eta = 1$.

[The results of the RG are confirmed by exact calculation using the so-called transfer matrix method.]

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2. *The Lifshitz Point*: (see Chaikin and Lubensky, p. 184) A number of materials, such as liquid crystals, are highly anisotropic and behave differently along directions parallel and perpendicular to some axis. An example is provided below. The d spatial dimensions are grouped into one parallel direction, $x_\parallel$ and $d-1$ perpendicular directions, $\mathbf{x}_\perp$. Consider a one-component field m subject to the Hamiltonian

$$\begin{aligned}
\beta H &= \beta H_0 + U, \\
\beta H_0 &= \int dx_\parallel \int d\mathbf{x}_\perp \left[\frac{K}{2}(\nabla_\parallel m)^2 + \frac{L}{2}(\nabla_\perp^2 m)^2 + \frac{t}{2}m^2 - hm \right], \\
U &= u \int dx_\parallel \int d\mathbf{x}_\perp m^4.
\end{aligned}$$

A Hamiltonian of this kind is realised in the theory of fluctuations in stacked fluid membranes — the smectic liquid crystal. [Note that βH depends on the *first* gradient in the $x_\parallel$ direction, and on the *second* gradient in the $\mathbf{x}_\perp$ directions.]

(a) Write βH_0 in terms of the Fourier transforms $m(q_\parallel, \mathbf{q}_\perp)$.

(b) Construct a renormalisation group transformation for βH_0 by rescaling distances such that $q'_\parallel = bq_\parallel$, $\mathbf{q}'_\perp = c\mathbf{q}_\perp$, and the field $m' = m/z$.

(c) Choose c and z such that $K' = K$ and $L' = L$. At the resulting fixed point calculate the eigenvalues y_t and y_h.

(d) Write down the relationship between the free energies $f(t,h)$ and $f(t',h')$ in the original and rescaled problems. Hence write the unperturbed free energy in the homogeneous form

$$f(t,h) = t^{2-\alpha} g_f(h/t^{\Delta}),$$

and identify the exponents α and Δ.

(e) How does the unperturbed zero-field susceptibility $\chi(t,0)$ diverge as $t \to 0$?

In the remainder of this problem set $h = 0$, and treat U as a perturbation.

(f) In the unperturbed Hamiltonian calculate the expectation value $\langle m(\mathbf{q}) m(\mathbf{q}') \rangle_0$, and the corresponding susceptibility $\chi(\mathbf{q})$, where $\mathbf{q} = (q_{\parallel}, \mathbf{q}_{\perp})$.

(g) Write the perturbation U in terms of the Fourier modes $m(\mathbf{q})$.

(h) Obtain the expansion for $\langle m(\mathbf{q}) m(\mathbf{q}') \rangle$ to first order in U, and reduce the correction term to a product of two-point expectation values.

(i) Write down the expression for $\chi(\mathbf{q})$ in the first order of perturbation theory, and identify the transition point t_c at first order in u. [Do not evaluate the integral explicitly.]

(j) Using RG, or any other method, find the upper critical dimension d_u for validity of the Gaussian exponents.