UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part III

REVISION PACK

EXPERIMENTAL AND THEORETICAL PHYSICS
Minor Topics: Phase Transitions

List of non-examinable topics:

Topological Phase transitions (chapter 5)

Epsilon Expansion

Lattice Gauge Theory

O(3) Quantum Rotors

Finite-size Scaling and Quantum Criticality

(21st March 2021)

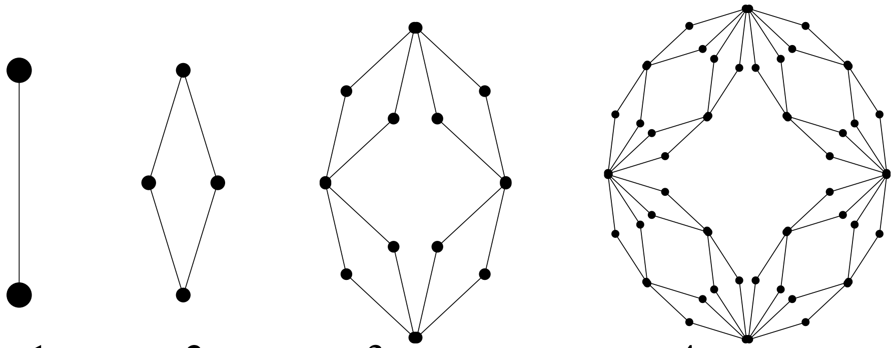
1 In one-dimension, the q-state Potts model is defined by the lattice Hamiltonian

$$\beta H = -K \sum_i \delta_{s_i, s_{i+1}} + g,$$

where the variables, s_i, can take only integer values from 1 to q, and g and $K > 0$ are constants.

(a) By integrating over the variables s_i at every other site, implement a real space renormalisation group procedure to obtain the exact recursion relations for K and g. [7]

(b) Show that $K^* = 0$ and $K^* = \infty$ are both fixed points of the Hamiltonian, and explore their stability. [6]



n=1 n=2 n=3 n=4

(c) The figure below shows the first four generations of a heirarchical lattice. Each generation, n, is obtained from generation n by replacing each bond by a 'diamond' of new bonds. Generalizing the real space renormalization group analysis above, show that, in the thermodynamic limit, $n \to \infty$, the Hamiltonian has a non-trivial fixed point defined by the condition, [5]

$$K^* = 2 \ln \left[\frac{q - 1 + e^{2K^*}}{q - 2 + 2e^{K^*}} \right].$$

Without explicit computation, sketch the resulting renormalisation group flows for the three fixed points. [2]

2 Outline concisely the conceptual basis of the *Renormalisation Group* (RG) method. [5]

In the Gaussian approximation, the Ginzburg-Landau Hamiltonian for the disordered phase of a 'smectic liquid crystal' takes the form

$$\beta H[m(\boldsymbol{x})] = \int \mathrm{d}x_\parallel \int \mathrm{d}^{d-1}\boldsymbol{x}_\perp \left[\frac{t}{2} m^2 + \frac{K}{2} (\nabla_\parallel m)^2 + \frac{L}{2} (\nabla_\perp^2 m)^2 - hm \right]$$

where $m(\boldsymbol{x})$ represents a one-component field depending on a d-dimensional set of coordinates $\boldsymbol{x} = (x_\parallel, \boldsymbol{x}_\perp)$, and the coefficients K, L, and t assume positive values.

(a) Transforming to the Fourier basis, reexpress the Hamiltonian $\beta H[m]$ in terms of the fields $m(q_\parallel, \boldsymbol{q}_\perp)$. [3]

(b) Construct a Renormalisation Group transformation for the Hamiltonian $\beta H[m]$ by (i) applying an anisotropic rescaling of the coordinates such that $q'_\parallel = b\, q_\parallel$ and $\boldsymbol{q}'_\perp = c\,\boldsymbol{q}_\perp$, and (ii) applying the field renormalisation $m' = m/z$. How do the parameters t, K, L, and h scale under the RG transformation? [10]

(c) For what values of c and z (as a function of b) do the parameters K and L remain fixed? For the remaining coefficients t and h, show that the corresponding Gaussian fixed point is associated with the exponents $y_t = 2$ and $y_h = (d+5)/4$ respectively. [3]

(d) By establishing the relationship between the free energies $f(t,h)$ and $f(t',h')$ of the original and rescaled Hamiltonians, show that the free energy assumes the homogeneous form

$$f(t,h) = t^{2-\alpha} g_f(h/t^{\Delta}).$$

Identify the exponents α and Δ. [4]

3 In the leading approximation, the influence of lattice compressibility on the ferromagnetic transition can be explored within the framework of the Ginzburg-Landau Hamiltonian

$$\beta H[m,\phi] = \int d^3\boldsymbol{x} \left[\frac{t}{2}m^2 + um^4 + vm^6 + \frac{K}{2}(\nabla m)^2 - hm + g\phi\, m^2 + \frac{c}{2}\phi^2\right],$$

where $\phi(\boldsymbol{x})$ denotes the (scalar) strain field, and the parameters u and v are both assumed positive.

(a) Integrating out strain field fluctuations $\phi(\boldsymbol{x})$, show that the partition function for the magnetisation field is controlled by the effective Hamiltonian [4]

$$\beta H_{\text{eff}}[m] = \int d^3\boldsymbol{x} \left[\frac{t}{2}m^2 + \left(u - \frac{g^2}{2c}\right) m^4 + vm^6 + \frac{K}{2}(\nabla m)^2 - hm\right].$$

(b) Working in the Landau theory approximation, by sketching the m dependence of the Landau Hamiltonian for different values of the parameters, describe *qualitatively* the magnetic phase diagram for $h = 0$. In particular, discuss what happens when $\frac{g^2}{2c} > u$. [4]

(c) When $\frac{g^2}{2c} = u$, obtain the average magnetisation $\bar{m}(t, h=0)$, $\bar{m}(t=0, h)$, and susceptibility $\chi(t, h=0) = \left.\frac{\partial \bar{m}}{\partial h}\right|_{h=0}$ within the Landau theory approximation. [6]

4 If we define a Hamiltonian $\beta H[\phi] = \beta H_0[\phi] + U[\phi]$ as the sum of a free theory $\beta H_0[\phi]$ and a perturbation $U[\phi]$, show that the renormalisation group (RG) transformation resulting from field integration over fast field fluctuations $\phi_>$ results in the following renormalised Hamiltonian for the slow field fluctuations $\phi_<$,

$$\beta H'[\phi_<] = -\mathcal{Z}_>^0 + \beta H_0[\phi_<] - \ln\langle e^{-U[\phi_<,\phi_>]}\rangle_>$$

where $\mathcal{Z}_>^0 = \int D\phi_> e^{-\beta H_0[\phi_>]}$ and $\langle \cdots \rangle_> = \frac{1}{\mathcal{Z}_>^0} \int D\phi_> \cdots e^{-\beta H_0[\phi_>]}$. [5]

The two-dimensional sine-Gordon theory describes a free scalar field $\phi(\boldsymbol{x})$ perturbed by a periodic potential,

$$\beta H[\phi] = \int d^2\boldsymbol{x} \left[\frac{K}{2}(\nabla\phi)^2 + g\cos(\lambda\phi)\right],$$

where $K > 0$.

(a) Treating the periodic potential as a perturbation of the free Gaussian theory, and applying the perturbative momentum shell RG, show that the renormalised Hamiltonian takes the form [2]

$$\beta H'[\phi_<] = -\mathcal{Z}_>^0 + \beta H_0[\phi_<] + \int d^2\boldsymbol{x}\, g\langle\cos[\lambda(\phi_<(\boldsymbol{x}) + \phi_>(\boldsymbol{x}))]\rangle_> + \mathcal{O}(g^2)$$

(b) Working to first order in g, show that, under the RG transformation, the parameters obey the scaling relations [7]

$$\begin{cases} K(b) = Kz^2b^{-4} \\ g(b) = gb^2 \exp\left[-\frac{\lambda^2}{4\pi K}(1 - b^{-1})\right] \\ \lambda(b) = \zeta\lambda \end{cases}$$

[*Your discussion should indicate the significance of the parameters z, b and ζ in the RG. For a free Gaussian theory, you may assume the identity $\langle e^{i\lambda\phi(\boldsymbol{x})}\rangle = e^{-\lambda^2\langle\phi^2(\boldsymbol{x})\rangle/2}$.*]

(c) Focusing on the fixed Hamiltonian $K(b) = K$ (i.e. $z = b^2$), it may be confirmed that $\lambda(b) = \lambda$. In this case, setting $b = e^\ell \simeq 1 + \ell + \cdots$, show that the differential recursion relations translate to the form

$$\frac{dg}{d\ell} = g\left(2 - \frac{\lambda^2}{4\pi K}\right).$$

Identify the fixed point and sketch the renormalisation group flow. Comment briefly on the physical implications of the result. [6]

5 In the restricted solid-on-solid model, the Hamiltonian of a rough surface is specified by

$$H = K \sum_{\langle \boldsymbol{l}\boldsymbol{m}\rangle} |h_{\boldsymbol{l}} - h_{\boldsymbol{m}}|^{\infty} \,,$$

where the discrete coordinates $\boldsymbol{l}$ and $\boldsymbol{m}$ each index the sites of a two-dimensional square lattice, and the height variable $h_{\boldsymbol{l}}$ can take positive and negative integer values. Here we have used the notation $\langle \boldsymbol{l}\boldsymbol{m}\rangle$ to indicate that the sum involves only neighbouring sites of the lattice.

(a) Considering βH, where $\beta = \dfrac{1}{k_{\mathrm{B}}T}$ with T the temperature, show that the height difference between neighbouring sites can only assume values of ± 1 or zero. [3]

(b) As a consequence, taking the boundary conditions to be periodic, show that the $N \times N$ site Hamiltonian may be recast in terms of the $2 \times N \times N$ variables $n_{\boldsymbol{l}\boldsymbol{m}} = h_{\boldsymbol{l}} - h_{\boldsymbol{m}}$ indexing the bonds between neighbouring sites. Explain why the sum of $n_{\boldsymbol{l}\boldsymbol{m}}$ around each square plaquette (i.e. unit cell boundary) of the lattice is constrained to be zero, i.e. defining $\hat{\boldsymbol{e}}_x = (1,0)$ and $\hat{\boldsymbol{e}}_y = (0,1)$, for each lattice site $\boldsymbol{l}$, [4]

$$n_{\boldsymbol{l},\boldsymbol{l}+\hat{\boldsymbol{e}}_x} + n_{\boldsymbol{l}+\hat{\boldsymbol{e}}_x,\boldsymbol{l}+\hat{\boldsymbol{e}}_x+\hat{\boldsymbol{e}}_y} + n_{\boldsymbol{l}+\hat{\boldsymbol{e}}_x+\hat{\boldsymbol{e}}_y,\boldsymbol{l}+\hat{\boldsymbol{e}}_y} + n_{\boldsymbol{l}+\hat{\boldsymbol{e}}_y,\boldsymbol{l}} = 0 \,.$$

(c) Imposing these constraints using the identity $\displaystyle\int_0^{2\pi} \frac{\mathrm{d}\theta}{2\pi} \mathrm{e}^{\pm in\theta} = \delta_{n0}$ for integer n, show that the partition function can be written as [6]

$$\mathcal{Z} = \left(\prod_{\boldsymbol{l}} \int_0^{2\pi} \frac{\mathrm{d}\theta_{\boldsymbol{l}}}{2\pi} \right) \exp\left\{ \sum_{\langle \boldsymbol{l}\boldsymbol{m}\rangle} \ln\left[1 + 2\mathrm{e}^{-\beta K} \cos(\theta_{\boldsymbol{l}} - \theta_{\boldsymbol{m}})\right] \right\} .$$

(d) At low temperatures (i.e. $\beta K \gg 1$), show that the system becomes equivalent to that of the classical two-dimensional XY spin model. Without resorting to detailed calculation, discuss the significance of this correspondence for the phase behaviour of the restricted solid-on-solid model? [7]

6 Under what circumstances are the low-energy degrees of freedom of systems described by *Goldstone modes*?

An approximately flat surface (i.e. with no 'overhangs') in d-dimensions can be described by its height $h(\boldsymbol{x})$ as a function of the remaining $d-1$ coordinates $\boldsymbol{x} = (x_1, \cdots, x_{d-1})$. If the surface has a tension σ, the Hamiltonian is simply $H = \sigma A$, where A denotes the total surface area

$$A = \int d^{d-1}\boldsymbol{x} \left[1 + (\nabla h)^2\right]^{1/2} .$$

(a) At suffiently low temperatures there will be only slow variations in h. By expanding the Hamiltonian to quadratic order in h, express the total partition function as a functional integral.

(b) By using a Fourier transformation to diagonize the quadratic Hamiltonian into normal modes $h_{\boldsymbol{q}}$ (known as capillary waves), show that the low-energy excitations are described by Goldstone modes. Identify the symmetry breaking responsible for the generation of the Goldstone modes.

(c) Obtain an expression for the correlation function of the height $\langle [h(\boldsymbol{x}) - h(\mathbf{o})]^2 \rangle$ in the form of an integral and, without evaluating it explicitly, comment on the form of your result in dimensions $d = 2$, 3 and 4.

7 Define *Goldstone Modes* and the describe the concepts of *long-range* and *quasi long-range* order.

The superfluid transition is accompanied by the condensation of the complex order parameter $\psi = \sqrt{\rho_s} e^{i\varphi}$. Below the transition temperature T_c, the superfluid density ρ_s becomes finite, and fluctuations of the phase field ϕ are governed by the effective Ginzburg-Landau Hamiltonian,

$$\beta H = \frac{1}{2} \int d^d \boldsymbol{x} \rho_s \left(\nabla \varphi\right)^2 .$$

By evaluating the spatial correlation function of the phase field $\langle \varphi(\boldsymbol{x}) \varphi(0) \rangle$, show that the correlation function of the complex order parameter in a *two-dimensional* film decays as a power law,

$$\langle \psi(\boldsymbol{x}) \psi(0) \rangle = \rho_s \left(\frac{a}{|\boldsymbol{x}|} \right)^{1/2\pi\rho_s} ,$$

where a represents a short-distance cut-off.

8 Outline briefly the basis of the Ginzburg-Landau phenomenology.

The d-dimensional Ginzburg-Landau Hamiltonian for a n-component order parameter $\boldsymbol{m}(\boldsymbol{x})$ is given by

$$\beta H = \int d^d \boldsymbol{x} \left[\frac{K}{2} \left(\nabla \boldsymbol{m}\right)^2 + \frac{t}{2} \boldsymbol{m}^2 + u \left(\boldsymbol{m}^2\right)^2 \right]$$

where t denotes the reduced temperature, and $u > 0$, $K > 0$.

(a) In the mean-field or saddle-point approximation, show that there is a spontaneous symmetry breaking at $t = 0$, and determine the average magnetisation $\bar{m}$ as a function of the reduced temperature for $t < 0$.

(b) Including both longitudinal and transverse fluctuations

$$\boldsymbol{m}(\boldsymbol{x}) = \left(\bar{m} + \phi_\ell(\boldsymbol{x})\right) \hat{\boldsymbol{e}}_\ell + \sum_{i=2}^{n} \phi_t^i(\boldsymbol{x}) \hat{\boldsymbol{e}}_i,$$

expand βH to quadratic order in the fields ϕ_ℓ and ϕ_t^i.

(c) Expressed in Fourier representation $\phi_t^i(\boldsymbol{q}) = \int d\boldsymbol{x} e^{-i\boldsymbol{q}\cdot\boldsymbol{x}} \phi_t^i(\boldsymbol{x})$, obtain an expression for the transverse correlation functions $\langle \phi_t^i(\boldsymbol{q}) \phi_t^j(\boldsymbol{q}') \rangle$. Comment on the implications of this result for the long-distance (i.e. $|\boldsymbol{x}| \to \infty$) behaviour of the transverse correlation function $\langle \phi_t^i(\boldsymbol{x}) \phi_t^j(0) \rangle$ for $t < 0$ and $t > 0$.

(d) Using quantum-classical mapping and the above saddle-point analysis, comment on the nature of excitations of the following quantum-field Hamiltonian in the case $n > 1$

$$\hat{H} = \int d^d\boldsymbol{x} \left[\frac{\hat{\boldsymbol{p}}^2}{2K} + \frac{K}{2} \left(\nabla \hat{\boldsymbol{m}}\right)^2 + \frac{t}{2} \hat{\boldsymbol{m}}^2 + u \left(\hat{\boldsymbol{m}}^2\right)^2 \right], \tag{1}$$

where $[\hat{m}_i(\boldsymbol{x}), \hat{p}_j(\boldsymbol{x}')] = i\delta_{ij}\hbar\delta(\boldsymbol{x} - \boldsymbol{x}')$ are canonically conjugate operators (i indexes the component). Find the spectrum of excitations in the case $t < 0$. Using quantum-classical mapping write down the following ground state expectation values:

(i) $\langle \hat{\boldsymbol{m}}(\boldsymbol{x}) \rangle_{\text{g.s.}}$,

(ii) $\langle \hat{\boldsymbol{m}}(\boldsymbol{x}) \hat{\boldsymbol{m}}(\mathbf{o}) \rangle_{\text{g.s.}}$,

(iii) $\langle e^{\hat{H}\tau} \hat{\boldsymbol{m}}(\boldsymbol{x}) e^{-\hat{H}\tau} \hat{\boldsymbol{m}}(\mathbf{o}) \rangle_{\text{g.s.}}$, where $\tau > 0$.

[*Note that* $\int \frac{d^d\boldsymbol{q}}{(2\pi)^d} \frac{e^{i\boldsymbol{q}\cdot\boldsymbol{x}}}{\boldsymbol{q}^2 + \xi^2} \sim e^{-|\boldsymbol{x}|/\xi}$.]

9 *Non-examinable* The two-dimensional XY-model is described by the continuum Hamiltonian

$$\beta H = \int d^2\boldsymbol{x} \frac{J}{2} \left(\nabla \theta\right)^2$$

where the field $\theta(\boldsymbol{x})$ is periodic in 2π. Defining the concept of a Goldstone mode, explain concisely why this Hamiltonian does not exhibit a phase of long-range order at any finite temperature.

(a) Imposing a field configuration involving a single vortex, show that the corresponding energy is given by

$$\beta E_{\text{vortex}} = \pi J \ln\left(\frac{L}{a}\right) + \beta E_{\text{core}},$$

where a denotes a short-scale cut-off or lattice spacing, and βE_{core} represents some unspecified core energy.

(b) Using this result, estimate the contribution to the partition function from the vortex configuration as a function of the effective temperature J^{-1}. What implications does this result have on the nature of long-range order in the two-dimensional XY model?

10 Explain the concepts of *Spontaneous Symmetry Breaking* and *Goldstone modes* in statistical mechanics.

The low-energy properties of a classical d-dimensional XY-Ferromagnet are described by the Ginzburg-Landau Hamiltonian

$$\beta H = \frac{\bar{K}}{2} \int d^d\boldsymbol{x} \ (\nabla\theta)^2$$

where the corresponding two-component magnetisation field $\boldsymbol{m}(\boldsymbol{x}) = \bar{m}(\cos\theta(\boldsymbol{x}), \sin\theta(\boldsymbol{x}))$ is assumed to be constant in magnitude.

(a) Taking the fluctuations of the magnetisation field to be small, i.e. $\theta(\boldsymbol{x}) \ll 2\pi$, use the rules of Gaussian functional integration to show that the correlation function takes the form

$$\langle\theta(\boldsymbol{x})\theta(0)\rangle = -\frac{|\boldsymbol{x}|^{2-d}}{(2-d)S_d\bar{K}} + \text{const.}$$

where S_d denotes the d-dimensional solid angle.

(b) Using this result, show that

$$\begin{aligned}\lim_{|\boldsymbol{x}|\to\infty} \langle\boldsymbol{m}(\boldsymbol{x})\cdot\boldsymbol{m}(0)\rangle &= m_0^2,\ d > 2\\ &= 0,\ d \le 2\end{aligned}$$

where m_0 denotes some non-zero constant. Comment on the implications of this result for the nature of long-range order in low dimensions.
[Note that, for a Gaussian distribution, $\langle\exp[\alpha\theta]\rangle = \exp[\alpha^2\langle\theta^2\rangle/2]$.]

11 Close to the critical point of a classical Ferromagnet, the singular part of the free energy assumes the homogeneous form

$$f(t,h) = t^{2-\alpha} g_f\left(\frac{h}{t^{\Delta}}\right)$$

where $t = |T - T_c|/T_c$ represents the reduced temperature, and h denotes the dimensionless magnetic field.

(a) Using this expression for the free energy, obtain the homogeneous form for the magnetisation. With this result, determine the scaling exponents of the magnetisation $m(t, h = 0) \sim t^{\beta}$ and $m(t = 0, h) \sim h^{1/\delta}$ as a function of the exponents α and Δ.

(c) Using the expression for the magnetisation, obtain the scaling exponent γ of the susceptibility $\chi(t) \sim t^{-\gamma}$ as a function of α and Δ.

(d) According to the hyperscaling hypothesis, close to the critical point, the correlation length assumes the homogeneous form

$$\xi(t,h) = t^{-\nu} g_{\xi}\left(\frac{h}{t^{\Delta}}\right).$$

Explain why this result is compatible with the hyperscaling identity $d\nu = 2 - \alpha$.

(e) What can you say about the behaviour of the energy gap in the corresponding *quantum* ferromagnet?

12 The Ising ferromagnet on a d-dimensional cubic lattice in a magnetic field is defined by the Hamiltonian

$$\beta H = -\frac{1}{2}\sum_{ij}\sigma_i J_{ij}\sigma_j - h\sum_i \sigma_i, \qquad \sigma_i = \pm 1$$

where $J_{ij} = J$ if sites i and j are neighbours, and $J_{ij} = 0$ otherwise.

(a) Show that the partition function, $\mathcal{Z} = \sum_{\{\sigma_i\}} \exp(-\beta H[\{\sigma_i\}])$ can be expressed as

$$\mathcal{Z} = C\int_{-\infty}^{\infty}\prod_i dm_i \exp\left[-\frac{1}{2}\sum_{ij} m_i [J^{-1}]_{ij} m_j + \sum_i \sigma_i (m_i + h)\right]$$

where C denotes a constant.

(b) By expressing J^{-1} in Fourier space, show that

$$[J^{-1}]_{ij} = \frac{1}{2J}\int_{B.Z.}\frac{d^d\boldsymbol{q}}{(2\pi)^d}\frac{e^{i\boldsymbol{q}\cdot(\boldsymbol{r}_i - \boldsymbol{r}_j)}}{\cos q_1 + \cdots \cos q_d},$$

where $\boldsymbol{r}_i$ denotes the position vector of site i, and the lattice spacing is taken to be unity. [The momentum space integration runs over the Brillouin zone.]

(c) In the vicinity of the transition temperature, only long-range correlations are important. Show that in this limit, the partition function can be approximated by the Landau free energy

$$\beta H = \int d^d\boldsymbol{x}\left[\frac{K}{2}(\nabla m)^2 + \frac{t}{2}m^2 + um^4 - hm\right].$$

Your answer should indicate the dependence of the coefficients K, t, and u on the microscopic parameters of the theory.

(d) Determine the dependence of the magnetic susceptibility and the heat capacity in the vicinity of the mean-field transition both above and below the transition temperature.

13 Explain the meaning of the *upper critical dimension*?

A tricritical point is described by the Ginzburg-Landau Hamiltonian

$$\beta H = \int d^d\boldsymbol{r}\left[\frac{t}{2}\eta^2 + \frac{v}{6}\eta^6 + \frac{K}{2}(\nabla\eta)^2\right]$$

where the scalar field η represents the order parameter, and $t = (T - T_c)/T_c$ denotes the reduced temperature.

(a) In the *mean-field approximation*, find the dependence of the order parameter η, Free energy, and heat capacity on the reduced temperature t.

(b) Expanding the Hamiltonian to *quadratic order* around the mean-field solution, show that, away from the critical point, the correlation function $\langle \eta(\boldsymbol{r})\eta(0)\rangle$ decays exponentially at large distances, and determine the dependence of the correlation length on the reduced temperature t.

(c) By determining the fluctuation correction to the mean-field Free energy, determine the corresponding correction to the specific heat.

(d) Using this result, employ the Ginzburg criterion to show that the upper critical dimension $d_u = 3$.

(e) Describe qualitatively how these results are modified if the number of components of the order parameter are increased.

$[-\int \frac{d^d\boldsymbol{q}}{(2\pi)^d}\frac{e^{i\boldsymbol{q}\cdot\boldsymbol{r}}}{\boldsymbol{q}^2+\xi^{-2}} \approx \frac{|\boldsymbol{r}|^{2-d}}{(2-d)S_d}, |\boldsymbol{r}| \ll \xi;\ \ \frac{\xi^{(3-d)/2}}{(2-d)S_d|\boldsymbol{r}|^{(d-1)/2}}\exp(-|\boldsymbol{r}|/\xi), |\boldsymbol{r}| \gg \xi.]$

14 To quadratic order, the Ginzburg-Landau Hamiltonian for an n-component field configuration $\boldsymbol{m}(\boldsymbol{x})$ takes the form

$$\beta H[\boldsymbol{m}] = \int d^d\boldsymbol{x}\left[\frac{K}{2}(\nabla \boldsymbol{m})^2 + \frac{t}{2}\boldsymbol{m}^2 - \boldsymbol{h}\cdot\boldsymbol{m}\right]$$

where, throughout, we assume that $t > 0$ and $K > 0$.

(a) Applying Kadanoff's momentum shell renormalisation group, integrate out the degrees of freedom $\boldsymbol{m}(\boldsymbol{q})$ where $\Lambda/b < |\boldsymbol{q}| < \Lambda$ and Λ denotes the ultraviolet momentum cut-off. As a result, show that the parameters scale as

$$\begin{aligned} K' &= Kb^{-d-2}z^2 \\ t' &= tb^{-d}z^2 \\ h' &= hz \end{aligned}$$

where z defines the coefficient by which the order parameter is renormalised.

(b) Identify the Gaussian fixed point and show that the singular contribution to the free energy density $f(t,h) \equiv -(\ln \mathcal{Z})/V$ takes the scaling form

$$f_{\text{sing.}}(t,h) = t^{d/2} g_f\left(\frac{h}{t^{1/2+d/4}}\right).$$

(c) Adding to the fixed Hamiltonian the perturbation

$$u_n \int d^d\boldsymbol{x}|\boldsymbol{m}(\boldsymbol{x})|^n$$

obtain the scaling exponents y_n (i.e. $u'_n = b^{y_n}u_n$), and show that Gaussian fixed point is stable in dimensions $d > 4$.

15 The one-dimensional lattice Ising ferromagnet is described by the microscopic Hamiltonian

$$\beta H = -\sum_{ij} J_{ij}\sigma_i\sigma_j - h\sum_i \sigma_i,$$

where the spins $\sigma_i = \pm 1$, h denotes the magnetic field, and the exchange interaction varies with separation between sites i and j as $J_{ij} = Je^{-\kappa|i-j|}$ with $\kappa \ll 1$.

(a) Show that the partition function can be expressed in the form

$$\begin{aligned}\mathcal{Z} &= C\sum_{\sigma_i=\pm 1}\int\prod_{k=1}^{N} dm_k \exp\left[-\sum_{ij} m_i[J^{-1}]_{ij}m_j + 2\sum_i \sigma_i m_i\right] \\ &= C\int_{-\infty}^{\infty}\prod_k dm_k \exp\left[-\sum_{ij} m_i[J^{-1}]_{ij}m_j + \sum_i \ln(2\cosh(2m_i + h))\right]\end{aligned}$$

where C represents some unspecified constant.

(b) For the long-ranged model defined above, show that

$$\mathcal{Z} = C\int_{-\infty}^{\infty}\prod_k dm_k \exp\left[-\sum_j\left(\frac{1}{2J\sinh\kappa}(m_j - m_{j+1})^2 + U(m_j)\right)\right],$$

where $U(m) = \tanh(\kappa/2)m^2/J - \ln[2\cosh(2m + h)]$.

(c) Taking the continuum limit, show that the classical partition function is isomorphic to the quantum partition function of a particle in a double well potential.

16 Describe in outline Kadanoff's momentum shell renormalisation group. Long-range interactions between spins can be described by adding a term

$$\frac{1}{2}\int d^d\boldsymbol{x}_1\int d^d\boldsymbol{x}_2 J(\boldsymbol{x}_1 - \boldsymbol{x}_2)\ \boldsymbol{m}(\boldsymbol{x}_1)\cdot\boldsymbol{m}(\boldsymbol{x}_2)$$

to the usual Ginzburg-Landau Hamiltonian

$$\beta H[\boldsymbol{m}(\boldsymbol{x})] = \int d^d\boldsymbol{x}\ \left[\frac{t}{2}\boldsymbol{m}^2 + \frac{K_2}{2}(\nabla\boldsymbol{m})^2\right], \qquad t > 0.$$

(a) Setting $J(\boldsymbol{x}) \propto 1/|\boldsymbol{x}|^{d+\sigma}$, show that the Hamiltonian assumes the Gaussian form

$$\beta H[\boldsymbol{m}(\boldsymbol{q})] = \int\frac{d^d\boldsymbol{q}}{(2\pi)^d}\left[\frac{t}{2} + \frac{K_2}{2}\boldsymbol{q}^2 + \frac{K_\sigma}{2}|\boldsymbol{q}|^\sigma\right]\boldsymbol{m}(\boldsymbol{q})\cdot\boldsymbol{m}(-\boldsymbol{q})$$

where K_σ denotes some (unspecified) constant of proportionality, and $\boldsymbol{m}(\boldsymbol{q}) = \int d^d\boldsymbol{x}\ \boldsymbol{m}(\boldsymbol{x})\ e^{i\boldsymbol{q}\cdot\boldsymbol{x}}$.

(b) Applying the momentum shell renormalisation group, construct the recursion relations for (t, K_2, K_σ). Show that K_σ is irrelevant for $\sigma > 2$. What is the fixed Hamiltonian in this case?

(c) For $\sigma < 2$, show that the spin rescaling factor must be chosen such that $K_\sigma = K'_\sigma$, in which case K_2 is irrelevant. What is the new fixed Hamiltonian?

END OF PAPER

1 (a) To implement the real space renormalisation, we must integrate out spins at every other site to obtain a Hamiltonian with half the number of sites and renormalised coupling constants. Using the identity

$$\sum_{s=1}^{q} e^{K(\delta_{s_1,s}+\delta_{s,s_2})+g} = e^g \begin{cases} q-1+e^{2K} & s_1 = s_2 \\ q-2+2e^K & s_1 \neq s_2 \end{cases} \stackrel{!}{=} e^{K'\delta_{s_1,s_2}+g'},$$

and comparing the cases $\sigma_1 = \sigma_2$ and $\sigma_1 \neq \sigma_2$, we have

$$e^{g'} = (q-2+2e^K)e^g, \qquad e^{K'} = \frac{q-1+e^{2K}}{q-2+2e^K}.$$

(b) Setting $x = e^{K^*}$, the fixe point equation is given by

$$x = \frac{q-1+x^2}{q-2+2x}.$$

Solving this equation, we find

$$2x = -(q-2) \pm \left[(q-2)^2 + 4(q-1)\right]^{1/2} = -(q-2) \pm q,$$

i.e. $x = 1$ or $x = 1-q$. The latter solution translates to an imaginary value of k is is therefore unphysical. The former translates to $K^* = 0$. Then, for $K \ll 1$,

$$K' \simeq \ln\left[\frac{q+2K+2K^2}{q+2K+K^2}\right] \simeq \frac{K^2}{q} \ll K,$$

showing that the fixed point is stable. Conversely, for $K \gg 1$, we have

$$e^{K'} \simeq \frac{1}{2}e^K, \qquad K' = K - \ln 2 < K$$

showing that it is unstable.

(c) For the heirarchical lattice, we have

$$\left(\sum_{s=1}^{q} e^{K(\delta_{\sigma_1,s}+\delta_{s,\sigma_2})+g}\right)^2 = e^{2g} \begin{cases} (q-1+e^{2K})^2 & \sigma_1 = \sigma_2 \\ (q-2+2e^K)^2 & \sigma_1 \neq \sigma_2 \end{cases} \stackrel{!}{=} e^{K'\delta_{\sigma_1,\sigma_2})+g'},$$

where

$$e^{g'} = (q-2+2e^K)^2 e^{2g}, \qquad e^{K'} = \left(\frac{q-1+e^{2K}}{q-2+2e^K}\right)^2.$$

For $q = 2$, this translates to the relation

$$e^{K'} = \cosh^2(K), \qquad K' = 2\ln\cosh(K).$$

From this result, we obtain a non-trivial unstable fixed point at $K^* = 1.28$ in addition to the now stable fixed point at $K^* = \infty$ and $K^* = 0$.

2 The divergence of the correlation length at a second order phase transition suggests that, in the vicinity of the transition, the microscopic length-scales are irrelevant. The critical behaviour is dominated by fluctuations that are statistically self-similar up to the length scale ξ. Self-similarity allows the gradual elimination of the correlated degrees of freedom at length scales $|\boldsymbol{x}| \ll \xi$, until one is left with the relatively simple uncorrelated degrees of freedom at the scale of the correlation length ξ. [5]

(a) In the Fourier representation the Hamiltonian takes the diagonal form

$$\beta H = \frac{1}{2}\int \frac{d^d\boldsymbol{q}}{(2\pi)^d} G^{-1}(\boldsymbol{q})|m(\boldsymbol{q})|^2 - hm(\boldsymbol{q}=0),$$

where the anisotropic propagator is given by [3]

$$G^{-1}(\boldsymbol{q}) = t + Kq_{\parallel}^2 + L\boldsymbol{q}_{\perp}^4.$$

(b) To implement the RG procedure, the first step is to apply a course-graining by integrating over the fast field fluctuations. Setting

$$m(\boldsymbol{q}) = \begin{cases} m_<(\boldsymbol{q}) & 0 < |q_{\parallel}| < \Lambda/b \text{ and } 0 < |\boldsymbol{q}|_{\perp} < \Lambda/c, \\ m_>(\boldsymbol{q}) & \Lambda/b < |q_{\parallel}| < \Lambda \text{ or } \Lambda/c < |\boldsymbol{q}|_{\perp} < \Lambda, \end{cases}$$

the fast fluctuations separate from the slow identically for the Gaussian Hamiltonian. As such, an integration over the fast fluctuations obtains

$$\mathcal{Z} = \mathcal{Z}_> \int Dm_< \exp\left[-\frac{1}{2}\int_0^{\Lambda/b}(\mathrm{d}q_{\parallel})\int_0^{\Lambda/c}(\mathrm{d}^{d-1}\boldsymbol{q}_{\perp})G^{-1}(\boldsymbol{q})|m_<(\boldsymbol{q})|^2 + hm_<(0)\right],$$

where the constant $\mathcal{Z}_>$ is obtained from performing the functional integral over $m_>$. Applying the rescaling $q'_{\parallel} = bq_{\parallel}$ and $\boldsymbol{q}'_{\perp} = c\boldsymbol{q}_{\perp}$, the cut-off in the domain of momentum integration is restored. Finally, applying the renormalisation $m'(\boldsymbol{q}) = m_<(\boldsymbol{q})/z$ to the Fourier field amplitudes, one obtains

$$\mathcal{Z} = \mathcal{Z}_> \int Dm'(\boldsymbol{q}')e^{-(\beta H)'[m'(\boldsymbol{q}')]},$$

where the renormalised Hamiltonian takes the form

$$(\beta H)' = \frac{1}{2}\int (\mathrm{d}^d\boldsymbol{q})b^{-1}c^{-(d-1)}z^2\left(t + Kb^{-2}{q'_{\parallel}}^2 + Lc^{-4}{\boldsymbol{q}'_{\perp}}^4\right)|m'(\boldsymbol{q}')|^2 - zhm'(0).$$

From the result, we obtain the renormalisation of the coefficients [10]

$$\begin{cases} t' = tb^{-1}c^{-(d-1)}z^2, \\ K' = Kb^{-3}c^{-(d-1)}z^2, \\ L' = Lb^{-1}c^{-(d+3)}z^2, \\ h' = hz. \end{cases}$$

(c) Choosing parameters $c = b^{1/2}$ and $z = b^{(d+5)/4}$ ensures that $K' = K$ and $L' = L$ and implies the scaling exponents $y_t = 2$, $y_h = (d+5)/4$. [3]

(d) From this result we obtain the renormalisation of the free energy density

$$f(t,h) = b^{-(d+1)/2} f(b^2 t, b^{(d+5)/4} h).$$

Setting $b^2 t = 1$, we can identify the exponents $2 - \alpha = (d+1)/4$ and $\Delta = y_h/y_t = (d+5)/8$. [4]

3 The Hamiltonian given in the question represents the canonical form of the Ginzburg-Landau Hamiltonian for a second order phase transition. In the Landau theory, the functional integral for the classical partition function is approximated by its value at the Hamiltonian minimum, viz. [2]

$$\mathcal{Z} \equiv e^{-\beta F} = \int Dm(\boldsymbol{x}) e^{-\beta H[m(\boldsymbol{x})]} \simeq \exp\left[-\min_{m(\boldsymbol{x})} \beta H[m(\boldsymbol{x})]\right].$$

For $K > 0$, the minimal Hamiltonian is given by $m(\boldsymbol{x}) = \bar{m}$, constant. In this approximation, the free energy density is given by $f = \frac{\beta F}{V} = \beta H[\bar{m}]$, where $\bar{m} + 4u\bar{m}^3 - h = 0$. In particular, for $h = 0$, the magnetisation acquires a non-zero expectation value when $t < 0$ with $\bar{m} = \sqrt{-t/4u}$. Similarly, for $t = 0$, the magnetisation varies as $m = (h/3u)^{1/3}$. From this result, one can infer a phase diagram in which a line of first order transitions along $h = 0$ terminates at the critical point $t = 0$. Finally, differentiating the condition on $\bar{m}$ with respect to $\bar{m}$, one obtains the susceptibility [4]

$$\chi(t, h=0) = \frac{\partial m}{\partial h}\Big|_{h=0} = \begin{cases} 1/t & t > 0 \\ -1/2t & t < 0. \end{cases}$$

[*Full credit will be given even if the specific heat is not derived.*]

(a) In the presence of the strain field, the partition function is given by

$$\mathcal{Z} = \int Dm\, D\phi\, e^{-\beta H[m,\phi]}.$$

Being Gaussian in ϕ, the integral may be performed exactly and obtains

$$\int D\phi e^{-\int d^3\boldsymbol{x}\left[\frac{c}{2}\phi^2 + g\phi m^2\right]} = \int D\phi e^{-\int d^3\boldsymbol{x}\left[\frac{c}{2}(\phi - \frac{gm^2}{c})^2 - \frac{g^2}{2c}m^4\right]} = \text{const.} \times e^{\int d^3\boldsymbol{x}\left[\frac{g^2}{2c}m^4\right]},$$

leading to the suggested reduction in the quartic coefficient. [4]

(b) While the quartic coefficient $u' = u - g^2/2c$ remains positive, the Landau theory continues to predict a second order transition at $h = t = 0$. However, when the sign is reversed, the Landau Hamiltonian ($h = 0$)

$$\psi(m) = \frac{t}{2}m^2 + u'm^4 + vm^6$$

develops additional minima. By sketching $\psi(m)$ for different parameter values, one may see that, for $u' < 0$ and $t = 0$ the (degenerate) global minimum lies at some non-zero value of $\bar{m}$ while, for t large, the global minimum lies a $\bar{m} = 0$. In between, there exists a line of first order transitions which merges with the line of second order critical points at the tricritical point $t = u' = 0$. $\big[$*By careful calculation, one may show that the first order boundary follows the line* $t = u'^2/2v$.$\big]$ [4]

(c) Near the tricritical point ($u' = 0$ and $h = 0$), one obtains

$$\frac{\partial \psi}{\partial m} = m(t + 6vm^4) = 0, \qquad \bar{m}(t, h = 0) = \begin{cases} 0 & t > 0, \\ (-t/6v)^{1/4} & t < 0, \end{cases}$$

implying an exponent $\beta = 1/4$. Similarly, for $t = 0$, one obtains

$$h = 6v\bar{m}^5, \qquad \bar{m}(t = 0, h) = (h/6v)^{1/5}$$

i.e. $\delta = 5$. Finally, for finite h and t, differentiating the defining equation for $\bar{m}$, one obtains the susceptibility [4]

$$\chi(t, h = 0) = \frac{\partial \bar{m}}{\partial h}\bigg|_{h=0} = (t + 30v\bar{m}^4)^{-1},$$

implying that $\chi \sim 1/|t|$ for $t < 0$ and $t > 0$. Thus we find the exponent $\gamma = 1$. [2]

4 Separating the field fluctuations into fast and slow degrees of freedom, $\phi(\boldsymbol{x}) = \phi_<(\boldsymbol{x}) + \phi_>(\boldsymbol{x})$,

$$\begin{aligned} \mathcal{Z} &= \int D\phi_< e^{-\beta H_0[\phi_<]} D\phi_> e^{-\beta H[\phi_>] - U[\phi_<, \phi_>]} \\ &= \mathcal{Z}^0_> \int D\phi_< e^{-\beta H_0[\phi_<]} \langle e^{-U[\phi_<, \phi_>]} \rangle_> \\ &= \mathcal{Z}^0_> \int D\phi_< e^{-\beta H_0[\phi_<]} + \ln \langle e^{-U[\phi_<, \phi_>]} \rangle_>. \end{aligned}$$

From this result, one obtains the required renormalised Hamiltonian. [5]

(a) Applying the perturbative expansion,

$$-\ln \langle e^{-U[\phi_<, \phi_>]} \rangle_> \simeq \langle U[\phi_<, \phi_>] \rangle_> + O(U^2).$$

to the sine-Gordon theory, one obtains the required expression for the Hamiltonian. [2]

(b) Integrating over the fast field fluctuations, [2]

$$\begin{aligned}\langle g\cos[\lambda(\phi_<(\boldsymbol{x})+\phi_>(\boldsymbol{x}))]\rangle_> &= g\mathrm{Re}\left[e^{i\lambda\phi_<(\boldsymbol{x})}\langle e^{i\lambda\phi_>(\boldsymbol{x})}\rangle_>\right] \\ &= ge^{-\lambda^2\langle\phi_>^2(\boldsymbol{x})\rangle/2}\cos(\lambda\phi_<(\boldsymbol{x}))\end{aligned}$$

Then, making use of the identity, [2]

$$\langle\phi_>^2(\boldsymbol{x})\rangle = \int_> \frac{d^2\boldsymbol{q}}{(2\pi)^d}\frac{1}{Kq^2} = \frac{1}{2\pi K}(1-b^{-1})$$

and applying the rescalings, [2]

$$\boldsymbol{q}' = \boldsymbol{q}b, \qquad \phi'(\boldsymbol{q}') = \phi_<(\boldsymbol{q})/z, \qquad \phi'(\boldsymbol{x}') = \phi_<(\boldsymbol{x})/\zeta,$$

one obtains the renormalised Hamiltonian [1]

$$\beta H'[\phi'] = \int\frac{d^2\boldsymbol{q}}{(2\pi)^2}\frac{K(b)}{2}\boldsymbol{q}'^2|\phi(\boldsymbol{q})|^2 + \int d^2\boldsymbol{x}'\, g(b)\cos\left[\lambda(b)\phi'(\boldsymbol{x}')\right]$$

where the coefficients are as stated.

(c) Using the expansion, [3]

$$\begin{aligned} g(\ell) &= g(0)e^{2\ell}\exp\left[-\frac{\lambda^2}{4\pi K}(1-e^{-\ell})\right] \\ g(0)+\ell\frac{dg}{d\ell}+\cdots &= g(0)\left[1+2\ell-\frac{\lambda^2}{4\pi K}\ell+\cdots\right]\end{aligned}$$

one recovers the required differential recursion relation. For $\lambda^2 > 8\pi K$, $g(\ell)$ dimishes under the RG and the system flows towards a free massless theory. Conversely, for $\lambda^2 < 8\pi K$, $g(\ell)$ grows under RG leading to a confined or massive theory. When $\lambda_*^2 = 8\pi K$, the Hamiltonian is fixed and the theory critical. [3]

5 (a) For $h_i = h_j$ the site energy of a link $\beta H_{ij} = 0$; for $h_i = h_j \pm 1$ $\beta H_{ij} = K$; and $\beta H_{ij} \to \infty$ otherwise. Therefore, the former $n_{ij} = h_i - h_j = 0, \pm 1$ are the only allowed field configurations. [3]

(b) Taking into account the constraint $n_{ij} = 0, \pm 1$, one may note that the sum of n_{ij} around a plaquette $\sum_{\mathrm{ij}\in\mathrm{plaquette}} n_{ij} = 0$. Such a constraint ensures that the sum of n_{ij} around any closed loop must vanish since any loop can be decomposed into a set of elementary plaquettes. [4]

(c) Then, making use of the identity given in the question to impose the constraint, the partition function may be written as [6]

$$\mathcal{Z} = \sum_{n_{ij}=0,\pm 1} e^{-K|n_{ij}|}\left(\prod_{\boldsymbol{i}}\int_0^{2\pi} d\theta_{\boldsymbol{i}} e^{i(n_{\boldsymbol{i},\boldsymbol{i}+\hat{e}_x}+n_{\boldsymbol{i}+\hat{e}_x,\boldsymbol{i}+\hat{e}_x+\hat{e}_y}+n_{\boldsymbol{i}+\hat{e}_x+\hat{e}_y,\boldsymbol{i}+\hat{e}_y}+n_{\boldsymbol{i}+\hat{e}_y,\boldsymbol{i}})\theta_{\boldsymbol{i}}}\right),$$

where the product runs over all lattice sites $\boldsymbol{i}$. Noting that each site $\boldsymbol{i}$ is associated with two bonds along direction $\hat{\boldsymbol{e}}_x$ and $\hat{\boldsymbol{e}}_y$, the partition function may be rearranged as

$$\begin{aligned}\mathcal{Z} &= \left(\prod_{\boldsymbol{i}}\int_0^{2\pi} d\theta_{\boldsymbol{i}}\right)\left[\sum_{n=0,\pm1} e^{-K|n|}e^{i(\theta_{\boldsymbol{i}}+\theta_{\boldsymbol{i}-\hat{\boldsymbol{e}}_y})n}\right]\left[\sum_{n=0,\pm1} e^{-K|n|}e^{i(\theta_{\boldsymbol{i}}+\theta_{\boldsymbol{i}-\hat{\boldsymbol{e}}_x})n}\right]\\ &= \left(\prod_{\boldsymbol{i}}\int_0^{2\pi} d\theta_{\boldsymbol{i}}\right)\exp\left[\ln(1+2e^{-K}\cos(\theta_{\boldsymbol{i}}+\theta_{\boldsymbol{i}-\hat{\boldsymbol{e}}_y}))+\ln(1+2e^{-K}\cos(\theta_{\boldsymbol{i}}+\theta_{\boldsymbol{i}-\hat{\boldsymbol{e}}_x}))\right].\end{aligned}$$

Finally, setting $\theta_{\boldsymbol{i}} \mapsto -\theta_{\boldsymbol{i}}$ on alternate lattice sites, one obtains

$$\mathcal{Z} = \left(\prod_{\boldsymbol{i}}\int_0^{2\pi} d\theta_{\boldsymbol{i}}\right)\exp\left[\sum_{\langle \boldsymbol{ij}\rangle}\ln(1+2e^{-K}\cos(\theta_{\boldsymbol{i}}-\theta_{\boldsymbol{j}}))\right].$$

(d) At low temperatures ($K \gg 1$), the logarithm may be expanded as

$$\mathcal{Z} = \left(\prod_{\boldsymbol{i}}\int_0^{2\pi} d\theta_{\boldsymbol{i}}\right)\exp\left[2e^{-K}\sum_{\langle \boldsymbol{ij}\rangle}\cos(\theta_{\boldsymbol{i}}-\theta_{\boldsymbol{j}})\right].$$

The latter can be identified as the partition function of a two-dimensional XY model with exchange constant $J = 2e^{-K}$. This correspondence allows us to infer that the proliferation of massless fluctuations of the fields $\theta_{\boldsymbol{i}}$ leads to a disordering of the system for any non-zero temperature, i.e. spatial correlations of the height degrees of freedom allow for divergent fluctuations. However, since the present system lies at the lower critical dimension, one can infer that the restricted solid on solid model exhibits a topological Kosterlitz-Thouless phase transition from a phase with power-law correlations of the order parameter to exponential correlations. [7]

6 (a,b) Expanding the expression for the area we obtain the partition function

$$\begin{aligned}\mathcal{Z} &= \int Dh(\boldsymbol{x})e^{-\beta H[h]},\\ \beta H &= \beta\sigma A = \beta\sigma\int d^{d-1}\boldsymbol{x}\left[1+\frac{1}{2}(\nabla h)^2+\cdots\right]\\ &= \frac{\beta\sigma}{2}\int(d\boldsymbol{q})\boldsymbol{q}^2|h(\boldsymbol{q})|^2+\cdots\end{aligned}$$

(c) Making use of the correlator

$$\langle h(\boldsymbol{q}_1)h(\boldsymbol{q}_2)\rangle = (2\pi)^{d-1}\delta^{d-1}(\boldsymbol{q}_1+\boldsymbol{q}_2)\frac{1}{\beta\sigma\boldsymbol{q}_1^2},$$

we obtain the correlator

$$\left\langle [h(\boldsymbol{x})-h(0)]^2\right\rangle = \int (d\boldsymbol{q}_1)(d\boldsymbol{q}_2)\left(e^{i\boldsymbol{q}_1\cdot\boldsymbol{x}}-1\right)\left(e^{i\boldsymbol{q}_2\cdot\boldsymbol{x}}-1\right)\langle h(\boldsymbol{q}_1)h(\boldsymbol{q}_2)\rangle$$
$$= \frac{4}{\beta\sigma}\int (d\boldsymbol{q})\frac{\sin^2(\boldsymbol{q}\cdot\boldsymbol{x})}{\boldsymbol{q}^2}.$$

By inspection of the integrand, we see that for $d \geq 4$, the integral is dominated by $|\boldsymbol{q}| \gg 1/|\boldsymbol{x}|$, and

$$\left\langle [h(\boldsymbol{x})-h(0)]^2\right\rangle \sim \text{const.}$$

In three dimensions, the integral is logarithmically divergent and

$$\langle [h(\boldsymbol{x})-h(0)]\rangle \sim \frac{1}{\beta\sigma}\ln|\boldsymbol{x}|.$$

Finally, in two dimensions, the integral is dominated by small $\boldsymbol{q}$ and

$$\langle [h(\boldsymbol{x})-h(0)]\rangle \sim |\boldsymbol{x}|.$$

This result shows that in dimensions less than 4, a surface constrained only by its tension is unstable due to long-wavelength fluctuations.

7 To obtain full marks, the answer to the first part of the question should involve an account of spontaneous symmetry breaking in systems with a continuous symmetry. Marks will be given for writing a generic expression for the Ginzburg-Landau free energy describing Goldstone fluctuations; an estimate of the correlation functions in dimensionality $d = 2$, 3 and 4; a definition of the lower critical dimension, and a statement of the Mermin-Wagner theorem. Additional marks will be given for the mention of examples. It is expected that the calculation of the Green function for a point charge in d dimensions can be performed with the use of Gauss' theorem.

The calculation itself is a simple subset of the first part of the problem. The idea is that, by virtue of solving the technical part of the question, the student can use more time for discussion in the first part of the problem.

Applying Gauss' theorem as in the notes, the real space representation of the propagator can be found directly,

$$\left\langle [\varphi(\boldsymbol{x})-\varphi(0)]^2\right\rangle = \frac{2}{(2-d)S_d}\frac{1}{\rho_s}\left(x^{2-d}-a^{2-d}\right),$$

where a is the ultraviolet cut-off. The logarithmic dependence in two-dimensions should be easy to extract. As for the numerical prefactor, if all else fails, it can be deduced from the answer given.

8 The phenomenology of Ginzburg-Landau theory is based on the divergence of the correlation length in the vicinity of a second-order phase transition. This implies that singular critical properties of the theory depend only on fundamental symmetry properties of the model and not on the microscopic details of the Hamiltonian. This include locality, translational or rotational invariance, and scale invariance.

(a) In the mean-field approximation, the average magnetisation takes the homogeneous form $\boldsymbol{m} = \bar{m}\hat{\boldsymbol{e}}_\ell$ where $\hat{\boldsymbol{e}}_\ell$ represents a unit vector along some arbitrary direction, and $\bar{m}$ minimises the free energy density

$$f(\bar{m}) = \frac{\beta H[\bar{m}]}{V} = \frac{t}{2}\bar{m}^2 + u\bar{m}^4.$$

Differentiating, we find that

$$\begin{aligned}\bar{m} &= 0,\ t > 0 \\ &= \sqrt{-t/4u},\ t < 0\end{aligned}$$

(b) Applying the expansion, and making use of the identities

$$\begin{aligned}(\nabla\boldsymbol{m})^2 &= (\nabla\phi_\ell)^2 + (\nabla\phi_t^i)^2 \\ \boldsymbol{m}^2 &= \bar{m}^2 + 2\bar{m}\phi_\ell + \phi_\ell^2 + (\phi_t^i)^2 \\ \left(\boldsymbol{m}^2\right)^2 &= \bar{m}^4 + 4\bar{m}^3\phi_\ell + 6\bar{m}^2\phi_\ell^2 + 2\bar{m}^2(\phi_t^i)^2 + O(\phi^3)\end{aligned}$$

we find

$$\beta H = \beta H[\bar{m}] + \int d^2\boldsymbol{x}\frac{K}{2}\left[(\nabla\phi_\ell)^2 + \xi_\ell^{-2}\phi_\ell^2 + (\nabla\phi_t^i)^2 + \xi_t^{-2}(\phi_t^i)^2\right]$$

where

$$\begin{aligned}\frac{K}{\xi_\ell^2} &= t,\ t > 0 \\ &= -2t,\ t < 0 \\ \frac{K}{\xi_t^2} &= t,\ t > 0 \\ &= 0,\ t < 0\end{aligned}$$

(c) Expressed in Fourier representation, the Hamiltonian is diagonal and the transverse correlation function is given by

$$\langle\phi_t^i(\boldsymbol{q})\phi_t^j(\boldsymbol{q}')\rangle = (2\pi)^d\delta^d(\boldsymbol{q}+\boldsymbol{q}')\delta_{ij}\frac{1}{K(\boldsymbol{q}^2+\xi_t^{-2})}$$

Turning to the real space correlation function, for $t < 0$

$$\langle\phi_t^i(\boldsymbol{x})\phi_t^j(0)\rangle = \int\frac{d^d\boldsymbol{q}}{(2\pi)^d}\frac{e^{i\boldsymbol{q}\cdot\boldsymbol{x}}}{K\boldsymbol{q}^2} \sim \frac{1}{K|\boldsymbol{x}|^{d-2}}.$$

In dimensions $d > 2$ the correlation function decays at large distances while in dimensions $d \leq 2$ the correlation function diverges. This is consistent with the Mermin-Wagner theorem which implies the destruction of long-range order due to Goldstone mode fluctuations.

(d) The quantum Hamiltonian maps directly onto βH in one dimension higher. The inverse transverse correlation length in βH gives the energy gap between the ground state and the first excited state in $\hat{H}$. It follows that the excitations are gapped for $t > 0$ and gapless for $t < 0$ and that the energy gap vanishes as $\sqrt{t}$ as t approaches zero from above.

For $t < 0$, the low-energy spectrum is determined by transverse fluctuations of the magnetisation. Working in Fourier space, we can write

$$\hat{H} = \int \frac{d^d\boldsymbol{k}}{(2\pi)^d}\left[\frac{|\hat{\boldsymbol{p}}|^2}{2K} + \frac{K}{2}\left(\boldsymbol{k}^2 + \xi_t^{-2}\right)|\hat{\boldsymbol{m}}|^2 + \mathcal{O}(\boldsymbol{m}^4)\right]. \tag{2}$$

Each $\boldsymbol{k}$-mode is a simple harmonic oscillator with frequency $\omega(\boldsymbol{k}) = \sqrt{\boldsymbol{k}^2 + \xi_t^{-2}}$. For $t < 0$, $\xi_t^{-1} = 0$ and the spectrum is gapless.

(i) $\langle \hat{\boldsymbol{m}}(\boldsymbol{x})\rangle_{\text{g.s.}} = \langle \boldsymbol{m}(\boldsymbol{x})\rangle_{\beta H_{d+1}} = \bar{m}$,

(ii)

$$\begin{aligned}
&\langle \hat{\boldsymbol{m}}(\boldsymbol{x}) \cdot \hat{\boldsymbol{m}}(\mathbf{o})\rangle_{\text{g.s.}} = \langle \boldsymbol{m}(\boldsymbol{x}, \tau = 0) \cdot \boldsymbol{m}(\mathbf{o}, 0)\rangle_{\beta H_{d+1}} = \\
&\bar{m}^2 + \langle \phi_l(\mathbf{o}, 0)\phi_l(\boldsymbol{x}, \tau = 0)\rangle_{\beta H_{d+1}} + (n-1)\langle \phi_t(\mathbf{o}, 0)\phi_t(\boldsymbol{x}, \tau = 0)\rangle_{\beta H_{d+1}} \\
&\sim \begin{cases} n\exp[-|\boldsymbol{x}|\sqrt{\frac{K}{t}}], & t > 0, \\ \frac{n}{K|\boldsymbol{x}|^{d-1}}, & t = 0, \\ \sqrt{\frac{-t}{4u}} + \frac{n-1}{K|\boldsymbol{x}|^{d-1}} + \exp[-|\boldsymbol{x}|\sqrt{\frac{K}{-2t}}], & t \leq 0, \end{cases}
\end{aligned}$$

(iii)

$$\begin{aligned}
&\langle e^{\hat{H}\tau}\hat{\boldsymbol{m}}(\boldsymbol{x})e^{-\hat{H}\tau}\hat{\boldsymbol{m}}(\mathbf{o})\rangle_{\text{g.s.}} = \langle \boldsymbol{m}(\boldsymbol{x}, \tau) \cdot \boldsymbol{m}(\mathbf{o}, 0)\rangle_{\beta H_{d+1}} = \\
&\bar{m}^2 + \langle \phi_l(\mathbf{o}, 0)\phi_l(\boldsymbol{x}, \tau)\rangle_{\beta H_{d+1}} + (n-1)\langle \phi_t(\mathbf{o}, 0)\phi_t(\boldsymbol{x}, \tau)\rangle_{\beta H_{d+1}} \\
&\sim \begin{cases} n\exp[-\sqrt{\boldsymbol{x}^2 + \tau^2}\sqrt{\frac{K}{t}}], & t > 0, \\ \frac{n}{K(\boldsymbol{x}^2+\tau^2)^{\frac{d-1}{2}}}, & t = 0, \\ \sqrt{\frac{-t}{4u}} + \frac{n-1}{K(\boldsymbol{x}^2+\tau^2)^{\frac{d-1}{2}}} + \exp[-\sqrt{\boldsymbol{x}^2 + \tau^2}\sqrt{\frac{K}{-2t}}], & t \leq 0. \end{cases}
\end{aligned}$$

9 Switching to the momentum basis

$$\theta(\boldsymbol{q}) = \int d^2\boldsymbol{r} e^{i\boldsymbol{q}\cdot\boldsymbol{r}}\theta(\boldsymbol{r}), \qquad \theta(\boldsymbol{r}) = \int \frac{d^2\boldsymbol{q}}{(2\pi)^2} e^{-i\boldsymbol{q}\cdot\boldsymbol{r}}\theta(\boldsymbol{q}),$$

the Hamiltonian takes the form

$$\beta H = \frac{J}{2}\int \frac{d^2\boldsymbol{q}}{(2\pi)^2}\boldsymbol{q}^2|\theta(\boldsymbol{q})|^2$$

According to this result there exist low energy massless fluctuations of the field θ known as a Goldstone modes. The influence of these fluctuations on the long-range order in the system can be estimated by calculating the autocorrelator.

In the momentum basis, the autocorrelator of phases takes the form

$$\langle\theta(\boldsymbol{q}_1)\theta(\boldsymbol{q}_2)\rangle = (2\pi)^2\delta^2(\boldsymbol{q}_1+\boldsymbol{q}_2)\frac{1}{J\boldsymbol{q}_1^2}.$$

from which we obtain the real space correlator

$$\begin{aligned}\left\langle(\theta(\boldsymbol{r})-\theta(\mathbf{o}))^2\right\rangle &= \int\frac{d^2\boldsymbol{q}}{(2\pi)^2}\frac{|1-e^{i\boldsymbol{q}\cdot\boldsymbol{r}}|^2}{J\boldsymbol{q}^2} = 4\int\frac{d^2\boldsymbol{q}}{(2\pi)^2}\frac{\sin^2(\boldsymbol{q}\cdot\boldsymbol{r})}{J\boldsymbol{q}^2}\\ &= \frac{1}{\pi J}\ln\left(\frac{|\boldsymbol{r}|}{a}\right)\end{aligned}$$

where a represents some lower length scale cut-off. From this result, we see that the correlation function decays as a power law in two-dimensions corresponding to quasi-long range order. This is in accord with the Mermin-Wagner theorem which states that the breaking of a spontaneous symmetry is accompanied by the existence of massless Goldstone modes which destroy long-range order in dimensions $d \leq 2$.

(a) A vortex configuration of unit charge is defined by

$$\partial\theta(\boldsymbol{r}) = \frac{1}{|\boldsymbol{r}|}\hat{\boldsymbol{e}}_r\times\hat{\boldsymbol{e}}_z$$

Substituting this expression into the effective free energy, we obtain the vortex energy

$$\beta E_{\text{vortex}} = \frac{J}{2}\int\frac{d^2\boldsymbol{r}}{\boldsymbol{r}^2} = \pi J\ln\left(\frac{L}{a}\right) + \beta E_{\text{core}}$$

where a represents some short-distance cut-off and βE_{core} denotes the core energy.

(b) According to the harmonic fluctuations of the phase field, long-range order is destroyed at any finite temperature. However, the power law decay of correlations is consistent with the existence of quasi-long range order. The condensation of vortices indicates a phase transition to a fully disordered phase. An estimate for this melting temperature can be obtained from the single vortex configuration. Taking into account the contribution of a single vortex configuration to the partition function we have

$$\mathcal{Z} \sim \left(\frac{L}{a}\right)^2 e^{-\beta E_{\text{vortex}}}$$

where the prefactor is an estimate of the entropy. The latter indicates a condensation of vortices at a temperature $J = 2/\pi$.

10 In a second order phase transition an order parameter grows continuously from zero. The onset of order below the transition is accompanied by a spontaneous symmetry breaking — the symmetry of the low temperature ordered phase is lower than the symmetry of the high temperature disordered phase. An example is provided by the classical ferromagnet where the appearance of net magnetisation breaks the symmetry $m \mapsto -m$. If the symmetry is continuous, the spontaneous breaking of symmetry is accompanied by the appearance of massless Goldstone mode excitations. In the magnet, these excitations are known as spin waves.

(a) Applying the rules of Gaussian functional integration, one finds that $\langle \theta(\boldsymbol{x}) \rangle = 0$, and the correlation function takes the form

$$G(\boldsymbol{x}, \boldsymbol{x}') \equiv \langle \theta(\boldsymbol{x})\theta(0) \rangle = -\frac{C_d(\boldsymbol{x})}{\bar{K}}, \qquad \nabla^2 C_d(\boldsymbol{x}) = \delta^d(\boldsymbol{x})$$

where C_d denotes the Coulomb potential for a δ-function charge distribution. Exploiting the symmetry of the field, and employing Gauss', $\int d\boldsymbol{x}\; \nabla^2 C_d(\boldsymbol{x}) = \oint dS \cdot \nabla C_d$, one finds that C_d depends only on the radial coordinate x, and

$$\frac{dC_d}{dx} = \frac{1}{x^{d-1} S_d}, \qquad C_d(x) = \frac{x^{2-d}}{(2-d)S_d} + \text{const.},$$

where $S_d = 2\pi^{d/2}/(d/2-1)!$ denotes the total d-dimensional solid angle.

(b) Using this result, one finds that

$$\left\langle [\theta(\boldsymbol{x}) - \theta(0)]^2 \right\rangle = 2\left[\langle \theta(0)^2 \rangle - \langle \theta(\boldsymbol{x})\theta(0) \rangle\right] \overset{|\boldsymbol{x}|>a}{=} \frac{2(|\boldsymbol{x}|^{2-d} - a^{2-d})}{\bar{K}(2-d)S_d},$$

where the cut-off, a is of the order of the lattice spacing. (Note that the case where $d = 2$, the combination $|\boldsymbol{x}|^{2-d}/(2-d)$ must be interpreted as $\ln |\boldsymbol{x}|$. This result shows that the long distance behaviour changes dramatically at $d = 2$. For $d > 2$, the phase fluctuations approach some finite constant as $|\boldsymbol{x}| \to \infty$, while they become asymptotically large for $d \leq 2$. Since the phase is bounded by 2π, it implies that long-range order (predicted by the mean-field theory) is destroyed.

Turning to the two-point correlation function of $\boldsymbol{m}$, and making use of the Gaussian functional integral, obtains

$$\langle \boldsymbol{m}(\boldsymbol{x}) \cdot \boldsymbol{m}(0) \rangle = \bar{m}^2 \text{Re} \left\langle e^{i[\theta(\boldsymbol{x}) - \theta(0)]} \right\rangle.$$

For Gaussian distributed variables $\langle \exp[\alpha\theta] \rangle = \exp[\alpha^2 \langle \theta^2 \rangle / 2]$.
We thus obtain

$$\langle \boldsymbol{m}(\boldsymbol{x}) \cdot \boldsymbol{m}(0) \rangle = \bar{m}^2 \exp\left[-\frac{1}{2} \langle [\theta(\boldsymbol{x}) - \theta(0)]^2 \rangle \right] = \bar{m}^2 \exp\left[-\frac{(|\boldsymbol{x}|^{2-d} - a^{2-d})}{\bar{K}(2-d)S_d} \right],$$

implying a power-law decay of correlations in $d = 2$, and an exponential decay in $d < 2$. From this result we find

$$\lim_{|\boldsymbol{x}| \to \infty} \langle \boldsymbol{m}(\boldsymbol{x}) \cdot \boldsymbol{m}(0) \rangle = \begin{cases} m_0^2 & d > 2, \\ 0 & d \leq 2. \end{cases}$$

11 (a) From the free energy, one obtains the magnetisation as

$$m(t, h) \sim \frac{\partial f}{\partial h} \sim t^{2-\alpha-\Delta} g_m(h/t^{\Delta}).$$

In the limit $x \to 0$, $g_m(x)$ is a constant, and $m(t, h = 0) \sim t^{2-\alpha-\Delta}$ (i.e. $\beta = 2 - \alpha - \Delta$). On the other hand, if $x \to \infty$, $g_m(x) \sim x^p$, and $m(t = 0, h) \sim t^{2-\alpha-\Delta}(h/t^{\Delta})^p$. Since this limit is independent of t, we must have $p\Delta = 2 - \alpha - \Delta$. Hence $m(t = 0, h) \sim h^{(2-\alpha-\Delta)/\Delta}$ (i.e. $\delta = \Delta/(2 - \alpha - \Delta) = \Delta/\beta$).

(b) From the magnetisation, one obtains the susceptibility

$$\chi(t, h) \sim \frac{\partial m}{\partial h} \sim t^{2-\alpha-2\Delta} g_\chi(h/t^{\Delta}) \Rightarrow \chi(t, h = 0) \sim t^{2-\alpha-2\Delta} \Rightarrow \gamma = 2\Delta - 2 + \alpha.$$

(c) Close to criticality, the correlation length ξ is solely responsible for singular contributions to thermodynamic quantities. Since $\ln \mathcal{Z}(t, h)$ is dimensionless and extensive (i.e. $\propto L^d$), it must take the form

$$\ln \mathcal{Z} = \left(\frac{L}{\xi} \right)^d \times g_s + \left(\frac{L}{a} \right)^d \times g_a$$

where g_s and g_a are non-singular functions of dimensionless parameters (a is an appropriate microscopic length). (A simple interpretation of this result is obtained by dividing the system into units of the size of the correlation length. Each unit is then regarded as an independent random variable, contributing a constant factor to the critical free energy. The number of units grows as $(L/\xi)^d$. The singular part of the free energy comes from the first term and behaves as

$$f_{\text{sing.}}(t, h) \sim \frac{\ln \mathcal{Z}}{L^d} \sim \xi^{-d} \sim t^{d\nu} g_f(t/h^{\Delta})$$

As a consequence, comparing with the homogeneous expression for the free energy, one obtains the Josephson identity

$$2 - \alpha = d\nu$$

(d) The energy gap is given by the inverse correlation length along the imaginary time direction

$$\Delta(t,h) = \xi_\tau^{-1}(t,h) \sim t^{z\nu},$$

where z is the dynamical scaling exponent that relates the divergence of the correlation length along the temporal and spatial directions. In case of isotropic behaviour $z = 1$.

12 (a) Applying the Hubbard-Stratonovich transformation,

$$\exp\left[\frac{1}{2}\sum_{ij}\sigma_i J_{ij}\sigma_j\right] = C\int_{-\infty}^{\infty}\prod_k dm_k \exp\left[-\frac{1}{2}\sum_{ij} m_i[J^{-1}]_{ij}m_j + \sum_i m_i\phi_i\right],$$

and summing over the spin configurations, we obtain

$$\mathcal{Z} = C\int_{-\infty}^{\infty}\prod_k dm_k \exp\left[-\frac{1}{2}\sum_{ij} m_i[J^{-1}]_{ij}m_j + \sum_i \ln\left(2\cosh(2(m_i+h))\right)\right].$$

To determine J^{-1} it is convenient to switch to the basis in which J is diagonal — reciprocal space. Defining the Fourier series

$$\sigma(\boldsymbol{q}) = \sum_n e^{i\boldsymbol{q}\cdot\boldsymbol{n}}\sigma_{\boldsymbol{n}}, \qquad \sigma_{\boldsymbol{n}} = \int_{-\pi}^{\pi}(d\boldsymbol{q})e^{-i\boldsymbol{q}\cdot\boldsymbol{n}}\sigma(\boldsymbol{q}),$$

we obtain

$$\frac{1}{2}\sum_{ij}\sigma_i J_{ij}\sigma_j = \int(d\boldsymbol{q})J(\boldsymbol{q})|\sigma(\boldsymbol{q})|^2, \qquad J(\boldsymbol{q}) = \frac{J}{2}\sum_{\hat{e}} e^{i\boldsymbol{q}\cdot\hat{e}}$$

where $\hat{e}$ denote the primitive lattice vectors. From this result, we obtain the expansion

$$J(\boldsymbol{q}) = J\sum_{d=1}^{D}\cos q_d \approx D - \frac{\boldsymbol{q}^2}{2} + \cdots$$

Inverting and applying the inverse Fourier transform, we obtain

$$\begin{aligned}
[J^{-1}]_{ij} &= \int(d\boldsymbol{q})\frac{e^{i\boldsymbol{q}\cdot(\boldsymbol{n}_i-\boldsymbol{n}_j)}}{J(\boldsymbol{q})} \approx \frac{1}{J}\int(d\boldsymbol{q})e^{i\boldsymbol{q}\cdot(\boldsymbol{n}_i-\boldsymbol{n}_j)}\left(D^{-1} + \frac{\boldsymbol{q}^2}{2D} + \cdots\right) \\
&= \frac{1}{DJ}\delta_{\boldsymbol{n}_i,\boldsymbol{n}_j} - \frac{1}{2D^2J}\nabla^2_{\boldsymbol{n}_i-\boldsymbol{n}_j} + \cdots
\end{aligned}$$

Applying this expansion, we obtain

$$\mathcal{Z} \approx C\int Dm \exp\left[-\frac{1}{2}\int d^d\boldsymbol{x}\left(\frac{m^2}{DJ} + \frac{1}{2D^2J}(\nabla m)^2\right) + \int d^d\boldsymbol{x}\ln\left(2\cosh(2(m+h))\right)\right].$$

Expanding the logarithm for small h and m we obtain

$$\mathcal{Z} = \int Dm e^{-\beta H},$$

where the effective Ginzburg-Landau Hamiltonian takes the form

$$\beta H = \int d^d\boldsymbol{x} \left(\frac{K}{2}(\nabla m)^2 + \frac{t}{2}m^2 + um^4 - hm \right)$$

with

$$K = \frac{1}{4D^2J}, \qquad t = \frac{1}{2DJ} - 1, \qquad u = \frac{1}{12}.$$

Setting $t = 0$, we establish the critical point at $J_c^{-1} \equiv T_c^{-1} < 2D$.

(b-d) An estimate of the mean-field properties of the Ginzburg-Landau Hamiltonian is straightforward and can be found in the lecture notes.

13 (a) In the mean-field approximation (i.e. η is spatially non-varying), by minimising the Free energy density, it is straightforward to show that

$$\bar{\eta} = \begin{cases} 0 & t > 0, \\ (-t/v)^{1/4} & t > 0. \end{cases}$$

$$\beta \bar{F} = \begin{cases} 0 & t > 0, \\ -|t|^{3/2}/3v^{1/2} & t < 0. \end{cases}$$

From this result it is easy to obtain the heat capacity,

$$C_{\text{mf}} = -T\frac{\partial^2 f}{\partial t^2} \approx -T_c\frac{\partial^2 f}{\partial t^2} = \begin{cases} 0 & t > 0, \\ (-vt)^{-1/2}T_c/4 & t < 0. \end{cases}$$

(b) Expanding the Hamiltonian to second order in the vicinity of the mean field solution, one finds

$$\beta H(\eta) - \beta H(\bar{\eta}) = \frac{K}{2}\int d^d\boldsymbol{r} \left[(\nabla \eta)^2 + \frac{\eta^2}{\xi^2} \right], \qquad \frac{1}{\xi^2} = \begin{cases} t/K & t > 0, \\ -4t/K & t < 0. \end{cases}$$

From this result it is straightforward to determine the asymptotic form of the correlation function using the formula given at the end of the question.

$$\langle \eta(0)\eta(\boldsymbol{r})\rangle = \frac{e^{-|\boldsymbol{r}|/\xi}}{KS_d|d-2||\boldsymbol{r}|^{d-2}}$$

This identifies ξ as the correlation length which diverges in the vicinity of the transition.

(c) Again, in the Gaussian approximation, the free energy and heat capacity are easily determined.

$$\beta F = \frac{1}{2}\int \frac{d^d\boldsymbol{q}}{(2\pi)^d} \ln\left[K(\boldsymbol{q}^2+\xi^{-2})\right], \qquad C_{\text{fl}} = C - C_{\text{mf}} \propto K^{-d/2}|t|^{d/2-2}$$

(d) Taking the results for the mean field and fluctuation contribution to the heat capacity, one obtains,

$$\frac{C_{\text{fl}}}{C_{\text{mf}}} \propto \frac{|t|^{(d-3)/2}}{\sqrt{K^d/v}}$$

from which one can identify the upper critical dimension as 3.

(e) Most important difference is the appearance of Goldstone modes due to massless fluctuations of the transverse degrees of freedom. This gives rise to a power law decay of the correlation function below T_c.

14 14 (a) This question is, to a large extent, bookwork. Part (a) involves direct application of the RG procedure:

Coarse-Grain: The first step of the RG involves the elimination of fluctuations at scales $a < |\boldsymbol{x}| < ba$ or Fourier modes with wavevectors $\Lambda/b < |\boldsymbol{q}| < \Lambda$. We thus separate the fields into slowly and rapidly varying functions, $\boldsymbol{m}(\boldsymbol{q}) = \boldsymbol{m}_>(\boldsymbol{q}) + \boldsymbol{m}_<(\boldsymbol{q})$, where

$$\boldsymbol{m}(\boldsymbol{q}) = \begin{cases} \boldsymbol{m}_<(\boldsymbol{q}) & 0 < |\boldsymbol{q}| < \Lambda/b, \\ \boldsymbol{m}_>(\boldsymbol{q}) & \Lambda/b < |\boldsymbol{q}| < \Lambda. \end{cases}$$

Since the Ginzburg-Landau functional is Gaussian, the partition function is separable in the modes and can be reexpressed in the form

$$\mathcal{Z} = \int D\boldsymbol{m}_<(\boldsymbol{q}) e^{-\beta H[\boldsymbol{m}_<]} \int D\boldsymbol{m}_>(\boldsymbol{q}) e^{-\beta H[\boldsymbol{m}_>]}.$$

More precisely, the latter takes the form

$$\mathcal{Z} = \mathcal{Z}_> \int D\boldsymbol{m}_<(\boldsymbol{q}) \exp\left[-\int_0^{\Lambda/b} \frac{d^d\boldsymbol{q}}{(2\pi)^d}\left(\frac{t+K\boldsymbol{q}^2}{2}\right)|\boldsymbol{m}_<(\boldsymbol{q})|^2 + \boldsymbol{h}\cdot\boldsymbol{m}_<(0)\right],$$

where $\mathcal{Z}_> = \exp[-(nV/2)\int_{\Lambda/b}^{\Lambda}(d^d\boldsymbol{q}/(2\pi)^d)\ln(t+K\boldsymbol{q}^2)]$. [Full credit does not require an evaluation of the functional integral over $\boldsymbol{m}_>$.]

Rescale: The partition function for the modes $\boldsymbol{m}_<(\boldsymbol{q})$ is similar to the original, except that the upper cut-off has decreased to Λ/b, reflecting the coarse-graining in resolution. The rescaling, $\boldsymbol{x}' = \boldsymbol{x}/b$ in real space, is equivalent to $\boldsymbol{q}' = b\boldsymbol{q}$ in momentum space, and restores the cut-off to the original value.

Renormalise: The final step of the RG is the renormalisation of the field, $\boldsymbol{m}'(\boldsymbol{x}') = \boldsymbol{m}_<(\boldsymbol{x}')/\zeta$. Alternatively, we can renormalise the Fourier modes according to $\boldsymbol{m}'(\boldsymbol{q}') = \boldsymbol{m}_<(\boldsymbol{q}')/z$, resulting in

$$\begin{aligned}\mathcal{Z} &= \mathcal{Z}_> \int D\boldsymbol{m}'(\boldsymbol{q}') e^{-\beta H'[\boldsymbol{m}'(\boldsymbol{q}')]},\\ \beta H' &= \int_0^{\Lambda} \frac{d^d\boldsymbol{q}'}{(2\pi)^d} b^{-d} z^2 \left(\frac{t + Kb^{-2}\boldsymbol{q}'^2}{2}\right) |\boldsymbol{m}'(\boldsymbol{q}')|^2 - z\boldsymbol{h}\cdot\boldsymbol{m}'(0).\end{aligned}$$

As a result of the RG procedure the set of parameters $\{K, t, h\}$ has transformed from to a new set

$$\begin{cases} K' = Kb^{-d-2}z^2, \\ t' = tb^{-d}z^2, \\ h' = hz. \end{cases}$$

The singular point $t = h = 0$ is mapped onto itself as expected. To make the fluctuations scale invariant at this point, we must ensure that the remaining parameter in the Hamiltonian K stays fixed. This is achieved by the choice $z = b^{1+d/2}$ which implies

$$\begin{cases} t' = b^2 t & y_t = 2, \\ h' = b^{1+d/2} h & y_h = 1 + d/2. \end{cases}$$

For the fixed point $t = t'$, K becomes weaker and the spins become uncorrelated — the high temperature phase.

(b) From these equations, we can predict the scaling of the Free energy

$$\begin{aligned} f_{\text{sing.}}(t,h) &= b^{-d} f_{\text{sing.}}(b^2 t, b^{1+d/2} h), \qquad b^2 t = 1,\\ &= t^{d/2} g_f(h/t^{1/2+d/4}). \end{aligned}$$

[This implies exponents: $2-\alpha = d/2$, $\Delta = y_h/y_t = 1/2 + d/4$, and $\nu = 1/y_t = 1/2$. Comparing with the results from the exact solution we can can confirm the validity of the RG.]

(c) At the fixed point $(t = h = 0)$ the Hamiltonian is scale invariant. By dimensional analysis $\boldsymbol{x} = b\boldsymbol{x}'$, $\boldsymbol{m}(\boldsymbol{x}) = \zeta\boldsymbol{m}'(\boldsymbol{x}')$ and

$$(\beta H)^* = \frac{K}{2} b^{d-2}\zeta^2 \int d\boldsymbol{x}' \ (\nabla \boldsymbol{m}')^2, \qquad \zeta = b^{1-d/2}.$$

For small perturbations

$$(\beta H)^* + u_p \int d\boldsymbol{x}\ |\boldsymbol{m}|^p \to (\beta H)^* + u_p b^d \zeta^p \int d\boldsymbol{x}'\ |\boldsymbol{m}'|^p.$$

Thus, in general $u_p \to u'_p = b^d b^{p-pd/2} = b^{y_p} u_p$, where $y_p = p - d(p/2-1)$, in agreement with our earlier findings that $y_1 \equiv y_h = 1 + d/2$ and $y_2 \equiv y_t = 2$. For the Ginzburg-Landau Hamiltonian, the quartic term scales with an exponent $y_4 = 4 - d$ and is therefore relevant for $d < 4$ and irrelevant for $d > 4$. Sixth order perturbations scale with an exponent $y_6 = 6 - 2d$ and is therefore irrelevant for $d > 3$.

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(a) The following equality can be confirmed by integrating out the variables m_i on the right hand side:

$$\exp\left[\sum_{ij} J_{ij}\sigma_i\sigma_j\right] = C\int \prod_{k=1}^{N} dm_k \exp\left[-\sum_{ij} m_i[J^{-1}]_{ij} m_j + 2\sum_i \sigma_i m_i\right].$$

The classical partition function $\mathcal{Z} = \sum_{\{\sigma_i\}} e^{-\beta H[\sigma_i]}$ is given by

$$\mathcal{Z} = C\int \prod_{k=1}^{N} dm_k \exp\left[-\sum_{ij} m_i[J^{-1}]_{ij} m_j + \sum_i \ln\left(2\cosh(2m_i + h)\right)\right].$$

(b) To determine $[J^{-1}]_{ij}$, we transform to Fourier space. In particular, for the model at hand, after some algebra, one finds that the eigenvalues of J_{ij} are given by

$$J(q) = \sum_{n=-\infty}^{\infty} e^{iqn} J e^{-\kappa|n|} = \frac{J}{c - b\cos q}$$

where $c = \coth\kappa$ and $b = 1/\sinh\kappa$. Making use of this result we obtain

$$\begin{aligned}[J^{-1}]_{ij} &= \int_{-\pi}^{\pi} \frac{dq}{2\pi}\frac{e^{-iq(n_i-n_j)}}{J(q)} \\ &= c,\ i = j \\ &= b/2,\ i \pm 1 = j\end{aligned}$$

Therefore

$$\mathcal{Z} = C\int \prod_k dm_k \exp\left[-\frac{b}{2J}\sum_i (m_i - m_{i+1})^2 - \sum_i U(m_i)\right]$$

where $U(m) = (c-b)m^2/J - \ln[2\cosh(2m+h)]$. In particular $c - b = \tanh(\kappa/2)$.

(c) For small m and h the effective free energy can be expanded as

$$U(m) = -\ln 2 + \frac{t}{2}m^2 + \frac{4}{3}m^4 - 2hm + \cdots$$

where $t/2 = \tanh(\kappa/2)/J - 2$. Evidently, at zero magnetic field, the effective potential $U(m)$ is quartic. For $t < 0$, the potential takes the form of a double well.

(d) The path integral for a particle in a potential well is given by

$$\mathcal{Z} \;=\; \int Dr(\tau) \exp\left[-\frac{1}{\hbar}\int_0^\infty d\tau' \left(\frac{m}{2}\dot{r}^2 + U(r)\right)\right]$$

By identifying r with m, and τ with x, the partition function of the Ising model is seen to be equivalent to the path integral of a particle in a double well potential where the inverse temperature β is equivalent to the length of the spin chain L.

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The divergence of the correlation length at a second order phase transition implies self-similarity of spatial correlations. This, in turn, implies that the form of the free energy remains invariant under coordinate rescaling. This invariance is exploited in the renormalisation group procedure: The scaling of the parameters of the Ginzburg-Landau Hamiltonian under coordinate rescaling allows an identification of the fixed theory and the exposes the nature of the critical point. Operationally, the renormalisation procedure is implemented in three steps described in detail in the question:

(a) Expressed in a Fourier representation

$$\boldsymbol{m}(\boldsymbol{x}) = \int \frac{d^d\boldsymbol{q}}{(2\pi)^d}\boldsymbol{m}(\boldsymbol{q})e^{i\boldsymbol{q}\cdot\boldsymbol{x}}$$

With this definition, the long-range coupling of the Hamiltonian takes the form

$$\frac{1}{2}\int \frac{d^d\boldsymbol{q}}{(2\pi)^d}J(\boldsymbol{q})\boldsymbol{m}(\boldsymbol{q})\cdot\boldsymbol{m}(-\boldsymbol{q}), \qquad J(\boldsymbol{q}) = \int d^d\boldsymbol{x} J(\boldsymbol{x})e^{i\boldsymbol{q}\cdot\boldsymbol{x}} = K_\sigma|\boldsymbol{q}|^\sigma$$

With this result we obtain the expression shown in the text.

(b) Coarse-Grain: The first step of the RG involves the elimination of fluctuations at scales $a < |\boldsymbol{x}| < ba$ or Fourier modes with wavevectors $\Lambda/b < |\boldsymbol{q}| < \Lambda$. Applied to the Gaussian model described in the text, the fields can be separated into slowly and rapidly varying functions, $\boldsymbol{m}(\boldsymbol{q}) = \boldsymbol{m}_>(\boldsymbol{q}) + \boldsymbol{m}_<(\boldsymbol{q})$, where

$$\boldsymbol{m}(\boldsymbol{q}) = \begin{cases} \boldsymbol{m}_<(\boldsymbol{q}) & 0 < |\boldsymbol{q}| < \Lambda/b, \\ \boldsymbol{m}_>(\boldsymbol{q}) & \Lambda/b < |\boldsymbol{q}| < \Lambda. \end{cases}$$

Since the Ginzburg-Landau functional is Gaussian, the partition function is separable in the modes and can be reexpressed in the form

$$\mathcal{Z} = \int D\boldsymbol{m}_<(\boldsymbol{q})e^{-\beta H[\boldsymbol{m}_<]}\int D\boldsymbol{m}_>(\boldsymbol{q})e^{-\beta H[\boldsymbol{m}_>]}.$$

More precisely, the latter takes the form

$$\mathcal{Z} = \mathcal{Z}_> \int D\boldsymbol{m}_<(\boldsymbol{q}) \exp\left[-\int_0^{\Lambda/b}\frac{d^d\boldsymbol{q}}{(2\pi)^d}\left(\frac{t + K_2\boldsymbol{q}^2 + K_\sigma|\boldsymbol{q}|^\sigma}{2}\right)|\boldsymbol{m}_<(\boldsymbol{q})|^2\right],$$

where $\mathcal{Z}_>$ represents some irrelevant constant.

Rescale: The partition function for the modes $\boldsymbol{m}_<(\boldsymbol{q})$ is similar to the original, except that the upper cut-off has decreased to Λ/b, reflecting the coarse-graining in resolution. The rescaling, $\boldsymbol{x}' = \boldsymbol{x}/b$ in real space, is equivalent to $\boldsymbol{q}' = b\boldsymbol{q}$ in momentum space, and restores the cut-off to the original value.

Renormalise: The final step of the RG is the renormalisation of the field, $\boldsymbol{m}'(\boldsymbol{x}') = \boldsymbol{m}_<(\boldsymbol{x}')/\zeta$. Alternatively, we can renormalise the Fourier modes according to $\boldsymbol{m}'(\boldsymbol{q}') = \boldsymbol{m}_<(\boldsymbol{q}')/z$, resulting in

$$\begin{aligned}\mathcal{Z} &= \mathcal{Z}_> \int D\boldsymbol{m}'(\boldsymbol{q}')e^{-\beta H'[\boldsymbol{m}'(\boldsymbol{q}')]},\\ \beta H' &= \int_0^{\Lambda}\frac{d^d\boldsymbol{q}'}{(2\pi)^d}b^{-d}z^2\left(\frac{t + K_2 b^{-2}\boldsymbol{q}'^2 + K_\sigma b^{-\sigma}|\boldsymbol{q}'|^\sigma}{2}\right)|\boldsymbol{m}'(\boldsymbol{q}')|^2.\end{aligned}$$

As a result of the RG procedure the set of parameters $\{t, K_2, K_\sigma\}$ has transformed from to a new set

$$\begin{cases} t' = tb^{-d}z^2, \\ K_2' = K_2 b^{-d-2}z^2, \\ K_\sigma' = K_\sigma b^{-d-\sigma}z^2. \end{cases}$$

Setting $K_2' = K_2$, the fluctuations are made scale invariant by the choice $z = b^{1+d/2}$ from which one obtains the scaling relations

$$\begin{cases} t' = b^2 t & y_t = 2, \\ K_\sigma' = K_\sigma b^{2-\sigma} & y_\sigma = 2 - \sigma. \end{cases}$$

Thus for $\sigma > 2$, the parameter K_σ scales to zero. In this case the fixed Hamiltonian is simply

$$\beta H^* = \int d^d\boldsymbol{x}\frac{K_2}{2}(\nabla\boldsymbol{m})^2$$

(b) For $\sigma < 2$ setting $K'_\sigma = K_\sigma$, $z = (\sigma + d)/2$ and one obtains

$$\begin{cases} t' = b^\sigma t & y_t = \sigma, \\ K'_2 = K_2 b^{\sigma-2} & y_2 = \sigma - 2. \end{cases}$$

In this case K_2 scales to zero and the fixed Hamiltonian takes the form

$$\beta H^* = \frac{1}{2} \int d^d\boldsymbol{x}_1 \int d^d\boldsymbol{x}_2 J(\boldsymbol{x}_1 - \boldsymbol{x}_2)\boldsymbol{m}(\boldsymbol{x}_1) \cdot \boldsymbol{m}(\boldsymbol{x}_2)$$