

# Circular Motion

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25<sup>th</sup> August, 2010

## 1 Introduction

In this lecture, we shall study uniform circular motion, which represents a departure from your AS-level studies on rectilinear kinematics and dynamics. In general, mechanics is very complex: it is impossible to study the full motion of real bodies without making simplifying assumptions and approximations<sup>1</sup>. These modelling conditions often reduces the full problem into certain constituent types, which can be solved using simple mathematics and occur frequently. More complicated behaviour can be constructed using these basic mathematical tools. During your time in Cambridge we hope to cover linear, circular, parabolic and simple harmonic motion.

Historically, the appreciation of circular motion was crucial in determining the motion of the planets, the law of gravity and spurred Newton to produce a coherent set of laws first enunciated in his *Philosophiæ Naturalis Principia Mathematica*. Realising that acceleration was a vector quantity, he derived a formula for the acceleration of a particle travelling in a circle. Though it is possible for the speed of such a particle to be constant, since the direction of the velocity (tangential to the radius) is changing, the particle must be accelerating. Using this formula he was able to reason with Kepler's third law<sup>2</sup>, that gravity must obey an inverse square law. Furthermore this new law was universal — true both for celestial bodies as well as on the Earth. Indeed, the *Principia* is one of the most important works ever published: it states the laws of motion, forming the basis of all of classical mechanics.

In addition to planetary motion, circular motion appears whenever there is a force perpendicular to the body's velocity. For example, charged particles in a magnetic field undergo helical motion (circular motion superimposed with uniform motion along the field lines); aeroplanes banking during a turn; a gear inside a mechanism; roller-coasters doing a loop-the-loop; electrons in atoms; . . .

Circular motion represents a departure from the problems you have encountered before, which were likely to be effectively one-dimensional or rectilinear. Here the particle is constrained to move at a given distance from an origin. The natural co-ordinates to describe such a particle would be in terms of the distance or radius  $r$  from the origin and the angle,

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<sup>1</sup>For example, during your school studies you will have assumed that bodies could be treated as point particles. Your lectures on rotational dynamics will demonstrate that you can decompose rigid body motion into motion of the centre of mass and internal rotational motion about the centre of mass.

<sup>2</sup>The period of planetary orbits varies as the 3/2 power of the size of the orbit.

$\theta$  it makes from a given axis. In mathematical jargon, this set of  $(r, \theta)$  is called plane polar co-ordinates and can be related to standard Cartesian co-ordinates by  $(x, y) \equiv (r \cos \theta, r \sin \theta)$ .

Examples denoted by a dagger † should be attempted in class and you will be given assistance by the demonstrators if necessary. The unmarked questions are additional examples without necessarily being harder, and those with a star \* are expected to be more difficult and require considerable effort.

## 2 Definitions

Given we are using a new co-ordinate system, we must construct an equivalent rotational language to displacement, velocity and acceleration, and other useful angular ideas. Suppose that a line is changing its orientation with time, and that it rotates through an angle  $\Delta\theta$  in a time interval  $\Delta t$ . The **average angular velocity**  $\omega_{\text{av}}$  during the interval is defined by

$$\omega_{\text{av}} = \frac{\Delta\theta}{\Delta t}, \quad (1)$$

and the **instantaneous angular velocity**  $\omega$  is defined by

$$\omega = \lim_{\delta t \rightarrow 0} \frac{\delta\theta}{\delta t} = \frac{d\theta}{dt}. \quad (2)$$

The angular velocity is also called the angular frequency for reasons that will become apparent. The units of  $\omega$  are inverse seconds ( $\text{s}^{-1}$ ), and so  $\omega$  represents the number of *radians* swept out per unit time. This definition directly analogous to that of normal velocity  $\mathbf{v} = \frac{d\mathbf{r}}{dt}$ . In what follows, we shall specialise to uniform circular motion, and so  $\omega$  does not change with time.

The **period**  $T$  for one revolution is the time taken to describe an angle  $2\pi$  radians, thus using Definition (1)

$$\Delta t = \frac{\Delta\theta}{\omega} \quad (3)$$

$$T = \frac{2\pi}{\omega}. \quad (4)$$

Since frequency  $f = 1/T$  is the reciprocal of period, we have

$$\omega = 2\pi f. \quad (5)$$

The particle undergoing circular motion at radius  $r$  travels a distance  $2\pi r$  in time  $T$  with speed  $v$  so we have

$$T = \frac{2\pi r}{v}, \quad (6)$$

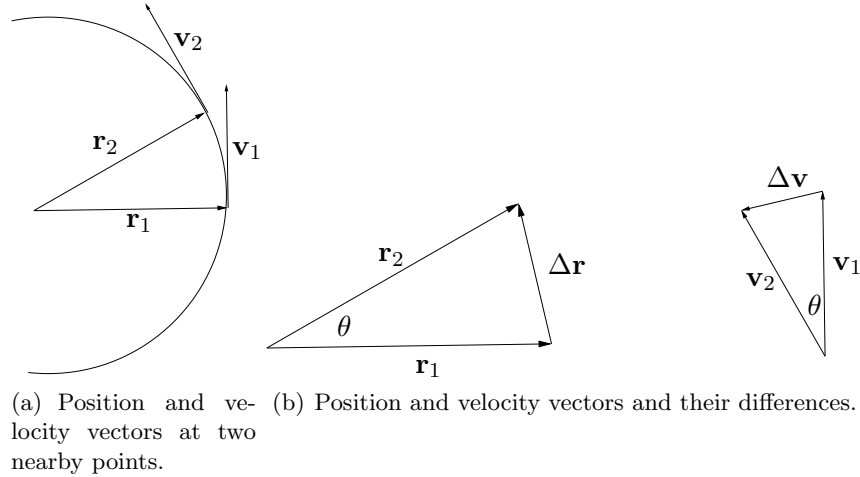
from which we find that

$$\boxed{v = \omega r.} \quad (7)$$

This result can also be found by directly differentiating the definition of arc-length.

### 3 Kinematics: Centripetal acceleration

As alluded to in the preamble, Newton realised that in order to keep bodies moving in a circle a force must be applied. Our familiar experience of being passengers in cars turning a bend suggest that the force is directed to the centre of the circle and thus called *centripetal*. Having a centripetal force implies, through Newton II, a radial acceleration. Let us calculate kinematically how this acceleration is related to the speed and radius of the circle.



The position and velocity vectors at two nearby times are shown in Figure 1(a). Their differences,  $\Delta \mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$  and  $\Delta \mathbf{v} = \mathbf{v}_2 - \mathbf{v}_1$ . The angle between the  $\mathbf{v}$ s is the same as the angle between the  $\mathbf{r}$ s, because each  $\mathbf{v}$  makes a right angle with the corresponding  $\mathbf{r}$ . The triangles in Figure 1(b) are therefore similar, so we have

$$\frac{|\Delta \mathbf{v}|}{v} = \frac{|\Delta \mathbf{r}|}{r} \quad (8)$$

where  $r = |\mathbf{r}|$  and  $v = |\mathbf{v}|$ . Dividing equation (8) through by  $\Delta t$  gives

$$\frac{1}{v} \left| \frac{\Delta \mathbf{v}}{\Delta t} \right| = \frac{1}{r} \left| \frac{\Delta \mathbf{r}}{\Delta t} \right| \quad (9)$$

$$\frac{|\mathbf{a}|}{v} = \frac{|\mathbf{v}|}{r} \quad (10)$$

$$a = \frac{v^2}{r}. \quad (11)$$

We have assumed that  $\Delta t$  is infinitesimal here, which allows us to get rid of the  $\Delta$ s in favour of instantaneous quantities. The instantaneous acceleration  $a$  is limiting value of the average acceleration. Using Equation (7) we may rewrite the acceleration as

$$\boxed{a = \frac{v^2}{r} = \omega^2 r.} \quad (12)$$

Note that this is not uniformly accelerated motion, because the value of  $\mathbf{a}$  is fixed only in magnitude, but not in direction. Note also that the direction of the acceleration is radially inwards.

### 3.1 Alternative derivations

1<sup>†</sup>. Calculate the average acceleration of a particle with speed  $v$  colliding elastically with a circular wall between any two collisions. Assume the container has radius  $r$  and take the angle of incidence of any collision as  $\theta$ . What happens as you vary  $\theta$ ?

2. Starting from the plane polar representation of the position vector

$$\mathbf{r} = \begin{pmatrix} r \cos \theta \\ r \sin \theta \end{pmatrix}, \quad (13)$$

find the velocity and acceleration vectors by direct differentiation with respect to time. Comment on their relative directions and magnitudes.

## 4 Dynamics

Let us now apply our acceleration formula (12) to analyse the motion of a car on a banked racing track and also over a hump-back bridge.

At what angle should a road be banked at a bend to avoid the need for a centripetal frictional force?

We treat the car as a point particle and resolve our force and acceleration vectors parallel to the line of greatest slope. Applying Newton II, we find

$$mg \sin \theta = m \left( \frac{v^2}{r} \cos \theta \right) \quad (14)$$

$$\tan \theta = \frac{v^2}{rg}. \quad (15)$$

The equation (which should *not* be learnt) shows that a bend of a given radius  $r$  can only be banked in this way for cars of a given speed  $v$ .

Now consider the car travelling over a bridge which is humped. Let us model the hump as part of an arc of a circle of radius  $r$ . What is the minimum speed of the car for it to leave the road? In general, applying Newton II radially inwards (vertically downwards) we find

$$mg - N = \frac{mv^2}{r}. \quad (16)$$

But since we require the condition that the car loses contact with the bridge, the normal reaction force,  $N$ , must go to zero. Hence Equation (16) reduces to

$$mg = \frac{mv^2}{r} \quad (17)$$

$$v^2 = gr, \quad (18)$$

which is the required condition. Any speed greater than  $\sqrt{gr}$  will cause the car to leave the bridge.

## 4.1 Further examples

3<sup>†</sup>. A mass hangs from a massless string of length  $l$ . Conditions have been set up so that the mass swings around in a horizontal circle, with the string making an angle  $\phi$  with the vertical. What is the angular frequency,  $\omega$ , of this motion?

4<sup>†</sup>. A person can be suspended against the inside of a hollow cylindrical room which is rotating about its central vertical axis even if there is no floor. If the radius of the cylinder is 2.0 m and the linear speed of the wall is  $8.0 \text{ m s}^{-1}$ , calculate the coefficient of friction which will prevent the person from falling.

5<sup>†</sup>. A stone of mass  $m$  attached to the end of a thread moves in a vertical circle of radius  $r$  so that the thread just remains fully extended at the top of the circle. Calculate the kinetic energy of the stone and the size of the tension force exerted by the string when

- (a) the stone is at its highest point
- (b) the string is horizontal
- (c) the stone is at its lowest point.

6\*. A point particle of mass  $m$  sits at rest on top of a frictionless hemisphere of mass  $M$ , which rests on a frictionless table. The particle is given a tiny kick and slides down the hemisphere. At what angle  $\theta$  (measured from the top of the hemisphere) does the particle lose contact with the hemisphere? \* First solve for the case  $M = \infty$ . \*\*\*\*\* In answering this question for  $m = M$ , it is sufficient for you to produce an equation that  $\theta$  must satisfy (it will be a cubic). However, for the special case of  $m = M$ , this equation can be solved without too much difficulty; find the angle in this case. *Hint: in addition to applying Newton's laws you might like to consider what quantities are conserved.*

## 4.2 Gravitational Examples

For those who have covered gravity:

7<sup>†</sup>. From the fact that the Moon orbits the Earth in approximately 28 days, estimate the distance of the Moon from the Earth's centre.

8. How high is a geostationary satellite above the Earth's surface. The radius of the Earth is 6400 km.

9\*. An Earth satellite of mass 20kg is in a circular orbit at a height small compared to the radius of the Earth.

If air resistance causes its total energy to decrease by 10kJ per revolution, what is the fractional change in its speed per revolution? Does the speed increase or decrease?